Introduction to Black-Scholes Theory

1 A random walk

We consider a random walk generated by the toss of a coin. If heads, we move a distance Δx along the x-axis, and if tails we move $-\Delta x$. We start at the origin at time zero, and toss 1 coin every Δt seconds until one second has elapsed. To summarize,

$$\begin{aligned} \text{step width} &= \Delta x \\ \text{number of coin tosses} &= n = \frac{1}{\Delta t} \\ \text{start position} &= 0 \text{ (the origin).} \end{aligned}$$

Here we want to take Δt as small as possible, with the aim of understanding the limiting case, as $\Delta t \downarrow 0$. Moreover, at the end of the experiment, we want the distribution to have a fixed variance. For this requirement we first determine conditions on Δx and Δt.

After 1 second, the probability of occupying the location $(2k-n)\Delta x$ is $\frac{1}{2^n}\binom{n}{k}$. Since the mean of the random variable is zero, the variance is given by

$$V = \sum_{k=0}^{n} (2k-n)^2 \Delta x^2 \cdot \frac{1}{2^k}\binom{n}{k}.$$

Simple calculations show that the variance satisfies

$$\begin{aligned} V &= \sum_{k=0}^{n} (2k-n)^2 \Delta x^2 \cdot \frac{1}{2^k}\binom{n}{k} \\ &\quad \vdots \\ &= \frac{\Delta x^2}{2^n}\left(n^2 2^n - 4n(n-1)2^{n-2}\right) \\ &= \Delta x^2 \cdot n = \frac{\Delta x^2}{\Delta t}. \end{aligned}$$

We can therefore set the variance by taking

$$\Delta x = \sigma\sqrt{\Delta t},$$

which yields the variance $= \sigma^2$ and standard deviation σ. From now we will write dZ to signify the random walk with $\sigma = 1$ (similarly, we will rewrite $\Delta t = dt$ and $\Delta x = dx$). Therefore we have a device for generating $n = 1/dt$ coin tosses within unit time, each of which changes our location by $\pm dx = \sqrt{dt}$. (We call the limit as $dt \to 0$ a gaussian process).

2 Modeling stock price changes

In this section we present a stock price model for building the foundation for the derivation of the Black-Scholes equation.

Suppose we have a stock worth S dollars. We assume that the stock, during the unit time, can be expressed as the sum of a gaussian process with variance σ and an expected return rate (comparisons with the real market have shown that such an assumption is often correct).

Hence during a small time dt the change in the stock price dS can be expressed:

$$dS = \mu S dt + \sigma S dZ. \tag{2.1}$$

Here we write

$$\begin{cases} S : \text{Stock price} \\ \mu : \text{Expected earnings percentage (a constant)} \\ \sigma : \text{Volatility (a constant)} \\ dZ : \text{Random walk (defined in the previous section)} \end{cases} \tag{2.2}$$

In this way, for a stock worth $S(t)$ dollars at time t, at time $t + dt$ we have

$$S(t + dt) = S(t) + dS = S(t) + \mu S(t)dt + \sigma S(t)dZ, \tag{2.3}$$

where dZ is the result of the coin toss that occurs each dt seconds:

$$dZ = \begin{cases} +dx = +\sqrt{dt} \\ -dx = -\sqrt{dt}. \end{cases} \tag{2.4}$$

3 The Black-Scholes equation

In this section we introduce the renowned Black-Scholes equation. We thus treat a method for pricing monetary articles, known as options (option pricing). Our targets are the following types of options.

1. European call option
   The right to buy a specified article for M dollars, at a specified date.

2. European put option
   The right to sell a specified article for M dollars, at a specified date.

Both the strike price and expiration date are specified beforehand. The problem is then to set the value v, of an option which expires at time T, for a stock that has price S at this time. Once we have expressed the option's value as a function $v(S,t)$ of the stock price S and time t, we will have rule for determining the value of the option.

Write the differential of v as dv, that of S as dS, and dt for the differential of T (here we must be careful with our calculations, as S also contains the elements of our random walk). Then

$$dv = v(S+ds, t+dt) - v(S,t),$$

and we write the Taylor expansion of the first term and collect the lowest order terms. Due to the relation $dx = \sqrt{dt}$, the we have the second order expansion:

$$dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial S}dS + \frac{1}{2}\frac{\partial^2 v}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 v}{\partial S^2}dS^2 + \frac{\partial^2 v}{\partial S \partial t}dSdt. \tag{3.1}$$

Using (2.1), we further expand dS in dt and dZ. Since $dZ = \pm dx$ is decided by the coin toss, and $dx = \sqrt{dt}$, one has $dZ^2 = dt$. Also, taking into account that $dZdt \to 0$, and $dt^2 \to 0$, collecting the first order terms in dt, one obtains the celebrated Ito's lemma:

$$dv = \left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2}\right)dt + \frac{\partial v}{\partial S}dS. \tag{3.2}$$

Lemma 3.1. (Ito's Lemma)
If a stock follows the Ito process $dS = \mu S dt + \sigma S dZ$, then

$$dv = \left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2}\right)dt + \frac{\partial v}{\partial S}dS.$$

From the above, the motions of the stock (2.1) and the option (3.2) are clear. With the random motions offsetting each other, the combination of these two yields rather stable properties. Then the arbitrage-free assumption asserts that secure assets can only mature with an expected rate of return μ. This is the Black-Scholes equation, which we now proceed to derive.

We put a total of Δ units of stock and an option of 1 unit into a portfolio Π, (here, as is customary, the symbol Δ describes an amount and is different from that of the previous section, which described an increment). That is,

$$\Pi = \Delta S - v,$$

from which the change in value of the portfolio is seen to be

$$d\Pi = \Delta dS - dv.$$

Inserting expression (3.2) for dv in the above and rearranging yields

$$d\Pi = \Delta dS - \left((\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2})dt + \frac{\partial v}{\partial S} dS \right).$$

Here if we always take $\Delta = \frac{\partial v}{\partial S}$, the random behavior from dS in the equation for $d\Pi$ disappears:

$$d\Pi = -\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} \right) dt.$$

An asset with the random property removed is called a *risk-free asset* (the operation of removing risk from the asset requires that $\Delta = \frac{\partial v}{\partial S}$ for all time, and is called *dynamic hedging*).

Under the arbitrage-free assumption, risk-free assets behave in the same way as an ordinary bank account with interest rate $r = \mu$, and so the following relation must hold:

$$r\Pi dt = -\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} \right) dt.$$

Since $\Pi = \Delta S - v$ and $\Delta = \frac{\partial v}{\partial S}$, using $\Pi = S\frac{\partial v}{\partial S} - v$ and rearranging yields the Black-Scholes equation:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = 0. \tag{3.3}$$

4 Boundary and terminal conditions

1. The call option

Since the value of the option at expiration must agree with any profit for selling the stock, we have the terminal condition

$$v(S,T) = \max(S - E, 0),$$

where E is the price at which the asset can be bought. Also, since the option is worthless whenever $S = 0$, we have the boundary condition

$$v(0,t) = 0.$$

Moreover, as $S \to \infty$, the option's value asymptotes:

$$v(S,t) \sim S - E \quad (S \to \infty).$$

Combining the above, the value of a *European call option* satisfies

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = 0 \\ v(S,T) = \max(S - E, 0) \\ v(0,t) = 0 \\ v(S,t) \sim S - E \quad (S \to \infty). \end{cases}$$

2. The put option

For the same reason as in the call option, the terminal condition here is

$$v(S,T) = \max(E - S, 0),$$

where E is the price at which the asset can be sold. If $S = 0$ at the time of expiration, the value of the option is E, but before this time, due to the arbitrage-free assumption, we must make the appropriate discount. The corresponding boundary condition is then

$$v(0,t) = Ee^{-r(T-t)}.$$

Also, as $S \to \infty$, the option value vanishes:

$$v(S,t) \sim 0 \quad (S \to \infty).$$

Combining the above, the value of the *European put option* satisfies

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = 0 \\ v(S,T) = \max(E - S, 0) \\ v(0,t) = Ee^{-r(T-t)} \\ v(S,t) \sim 0 \quad (S \to \infty). \end{cases}$$

In the above, we have used the following idea for determining the boundary condition. Let r be a simple interest rate for the term. Then the boundary condition at $S = 0$ is explained as a compound interest computation:

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r.$$

In particular, if we assume that we purchase the option at a time t between 0 and T, then the remaining time $T - t$ becomes a risk-free investment and, due to the arbitrage-free assumption, we need to deduct the accumulated interest.