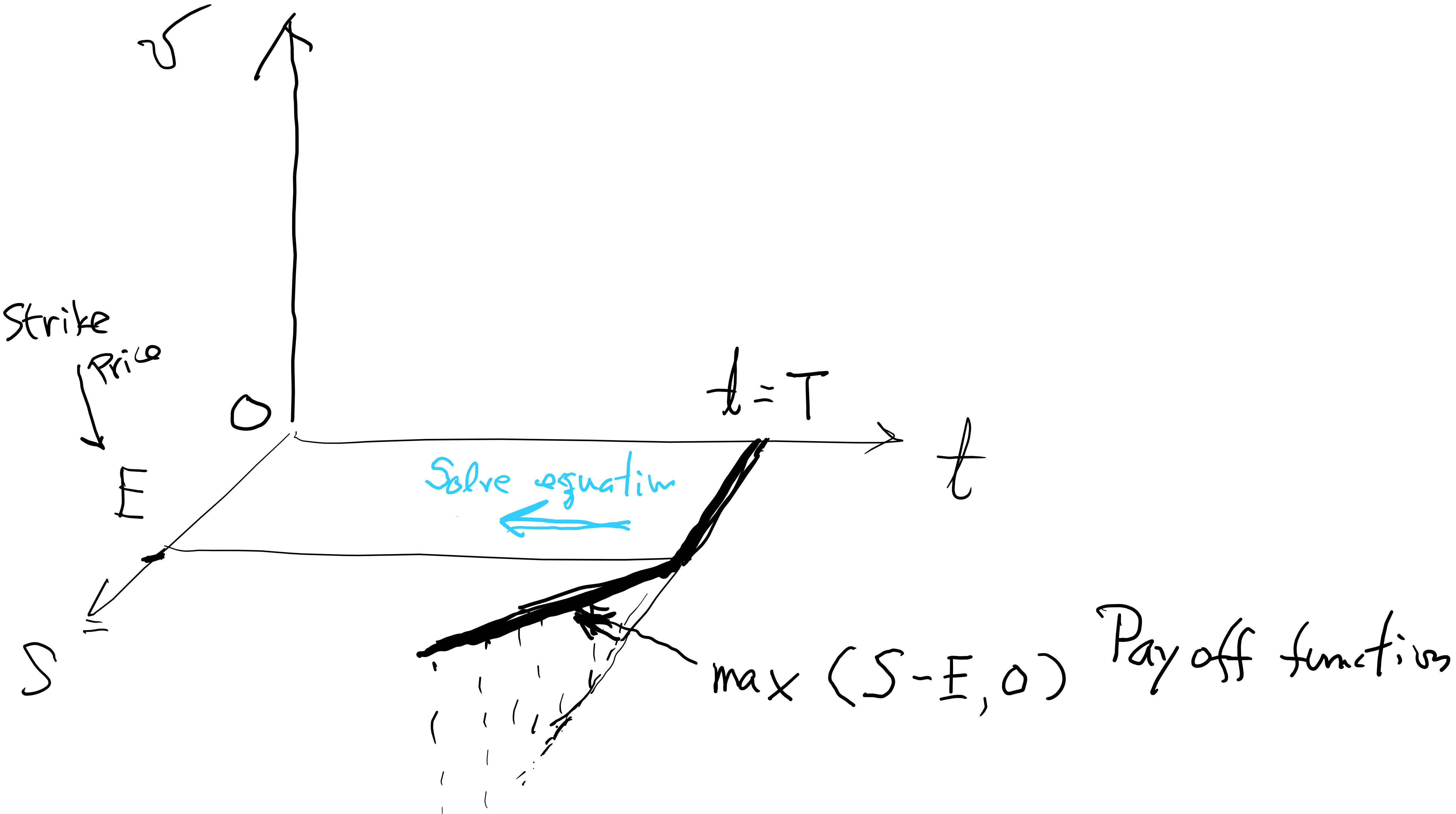
Variable transform from Black Scholes into heat equation

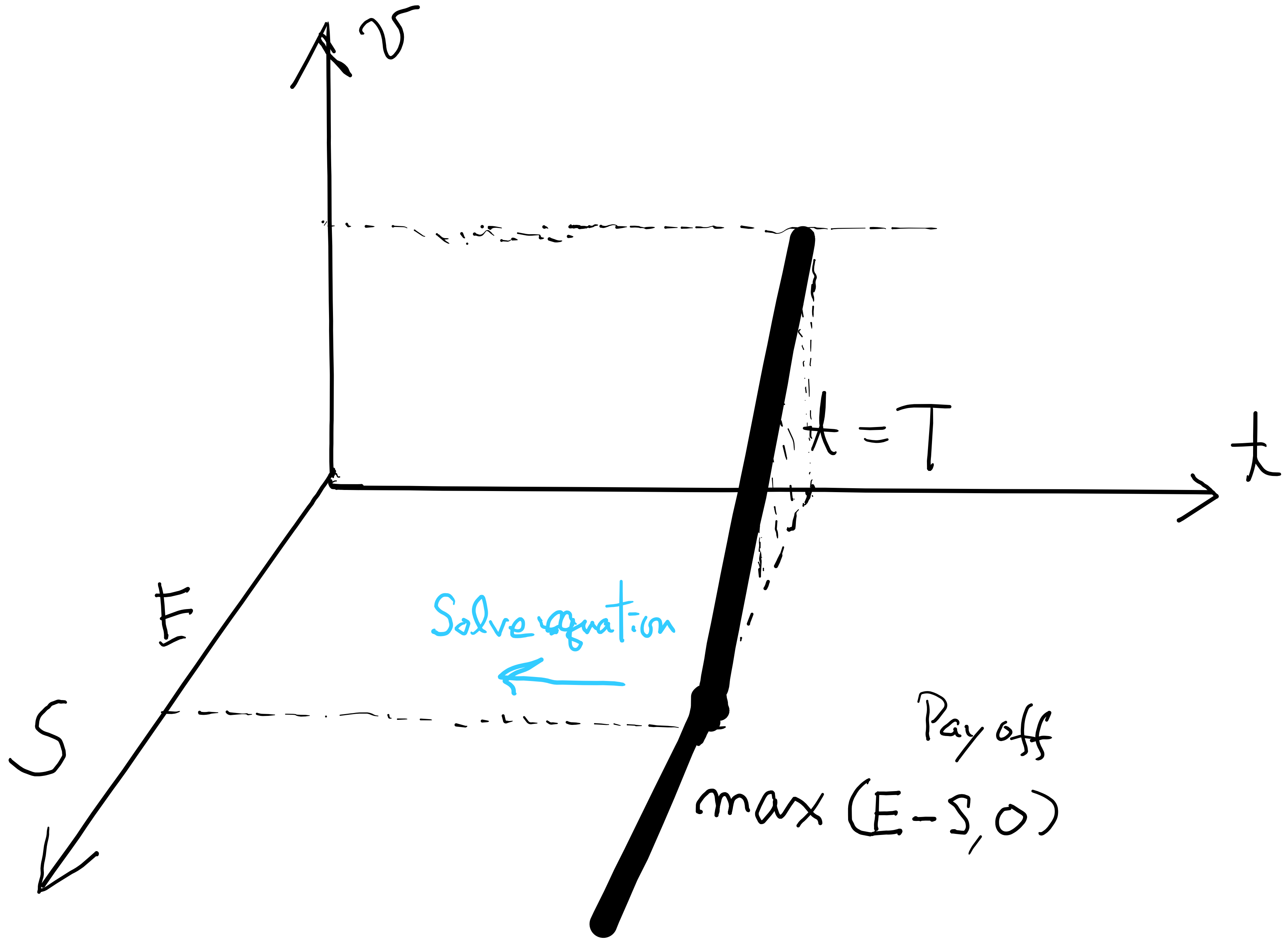
BS equation (call option case)

$$
\begin{cases}
\dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 v}{\partial S^2} + rS\dfrac{\partial v}{\partial S} - rv = 0 \qquad (S,t) \in (0,\infty) \times (0,\infty) \\
v(S,T) = \max(S-E,0) \\
v(0,t) = 0 \\
v(S,t) \sim S - E \ \ (S \to \infty)
\end{cases}
$$


V
Strike Price
O
E
S
t = T
t
Solve equation
max (S-E, 0)
Payoff function

Put option

$$
\begin{cases}
\dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 v}{\partial S^2} + rS\dfrac{\partial v}{\partial S} - rv = 0 \qquad (S,t) \in (0,\infty) \times (0,\infty) \\
v(S,T) = \max(E - S, 0) \\
v(0,t) = Ee^{-r(T-t)} \\
v(S,t) \sim 0 (S \to \infty)
\end{cases}
$$



V
t = T
t
E
S
Solve equation
Payoff
max (E-S,0)

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial S^2}+rS\frac{\partial v}{\partial S}-rv=0$$

- $\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial S^2}+rS\frac{\partial v}{\partial S}-rv=0$

- Put $y=\log(\frac{S}{E})$ and note that $\frac{\partial y}{\partial S}=\frac{1}{S}$
- $\frac{\partial v}{\partial S}=\frac{\partial v}{\partial y}\frac{\partial y}{\partial S}=\frac{\partial v}{\partial y}\frac{1}{S}\Leftrightarrow \mathrm{S}\frac{\partial v}{\partial S}=\frac{\partial v}{\partial y}$
- $\frac{\partial^2 v}{\partial S^2}=\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial S}\right)=\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial y}\frac{1}{S}\right)=\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial y}\right)\frac{1}{S}+\frac{\partial v}{\partial y}\frac{\partial}{\partial S}\left(\frac{1}{S}\right)$

$$=\frac{\partial}{\partial y}\frac{\partial y}{\partial S}\left(\frac{\partial v}{\partial y}\right)\frac{1}{S}-\frac{\partial v}{\partial y}\frac{1}{S^2}=\frac{\partial^2 v}{\partial y^2}\frac{1}{S^2}-\frac{\partial v}{\partial y}\frac{1}{S^2}$$

- Put $y = \log(\frac{S}{E})$ and note that $\frac{\partial y}{\partial S} = \frac{1}{S}$
- $\frac{\partial v}{\partial S} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial S} = \frac{\partial v}{\partial y}\frac{1}{S} \Leftrightarrow \mathrm{S}\frac{\partial v}{\partial S} = \frac{\partial v}{\partial y}$
- $\frac{\partial^2 v}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial v}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial v}{\partial y}\frac{1}{S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial v}{\partial y}\right)\frac{1}{S} + \frac{\partial v}{\partial y}\frac{\partial}{\partial S}\left(\frac{1}{S}\right)$

  $= \frac{\partial}{\partial y}\frac{\partial y}{\partial S}\left(\frac{\partial v}{\partial y}\right)\frac{1}{S} - \frac{\partial v}{\partial y}\frac{1}{S^2} = \frac{\partial^2 v}{\partial y^2}\frac{1}{S^2} - \frac{\partial v}{\partial y}\frac{1}{S^2}$

- $\Leftrightarrow S^2\frac{\partial^2 v}{\partial S^2} = \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}$

- Since $\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}) + r\frac{\partial v}{\partial y} - rv = 0$, then from previous calculation, we have

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial v}{\partial y} - rv = 0,$$

$$(y, t) \in (-\infty, \infty) \times (0, \infty)$$

Pay off function

- Pay off function: Call option case

$$v(S,T) = \max(S - E, 0) \Rightarrow v(y,T) = \max(E(e^y - 1), 0)$$

- Pay off function: Put option case

$$v(S,T) = \max(S - E, 0) \Rightarrow v(y,T) = \max(E(1 - e^y), 0)$$

Boundary condition at $\lim\limits_{y \longrightarrow \pm\infty} v(y,t)$

- Call case $\lim\limits_{y \longrightarrow -\infty} v(y,t) = 0$

  and $v(y,t) \sim E(e^y - 1)$ $\quad y \longrightarrow \infty$

- Put case $v(y,t) \sim Ee^{-r(T-t)}$ $\quad y \longrightarrow -\infty$

  and $\lim\limits_{y \longrightarrow \infty} v(y,t) = 0$

Further transform

We got,

- $\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial v}{\partial y} - rv = 0, \quad (y, t) \in (-\infty, \infty) \times (0, \infty)$

- Again, we put

$$x = y + \left(r - \frac{1}{2}\sigma^2\right)(T - t), \quad \tau = T - t$$

- $x = y + \left(r - \frac{1}{2}\sigma^2\right)(T - t), \quad \tau = T - t$
- $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial y} = \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial y^2}$
- $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial \tau} \frac{\partial \tau}{\partial t} = -\left(r - \frac{1}{2}\sigma^2\right) \frac{\partial v}{\partial x} - \frac{\partial v}{\partial \tau}$
- We have
- $-\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} - rv = 0,$

- We have
- $-\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2}-rv=0,$

- Multiply $e^{r\tau}$ both side of above, and put $u=ve^{r\tau}$ then
$$\frac{\partial u}{\partial t}=\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$$