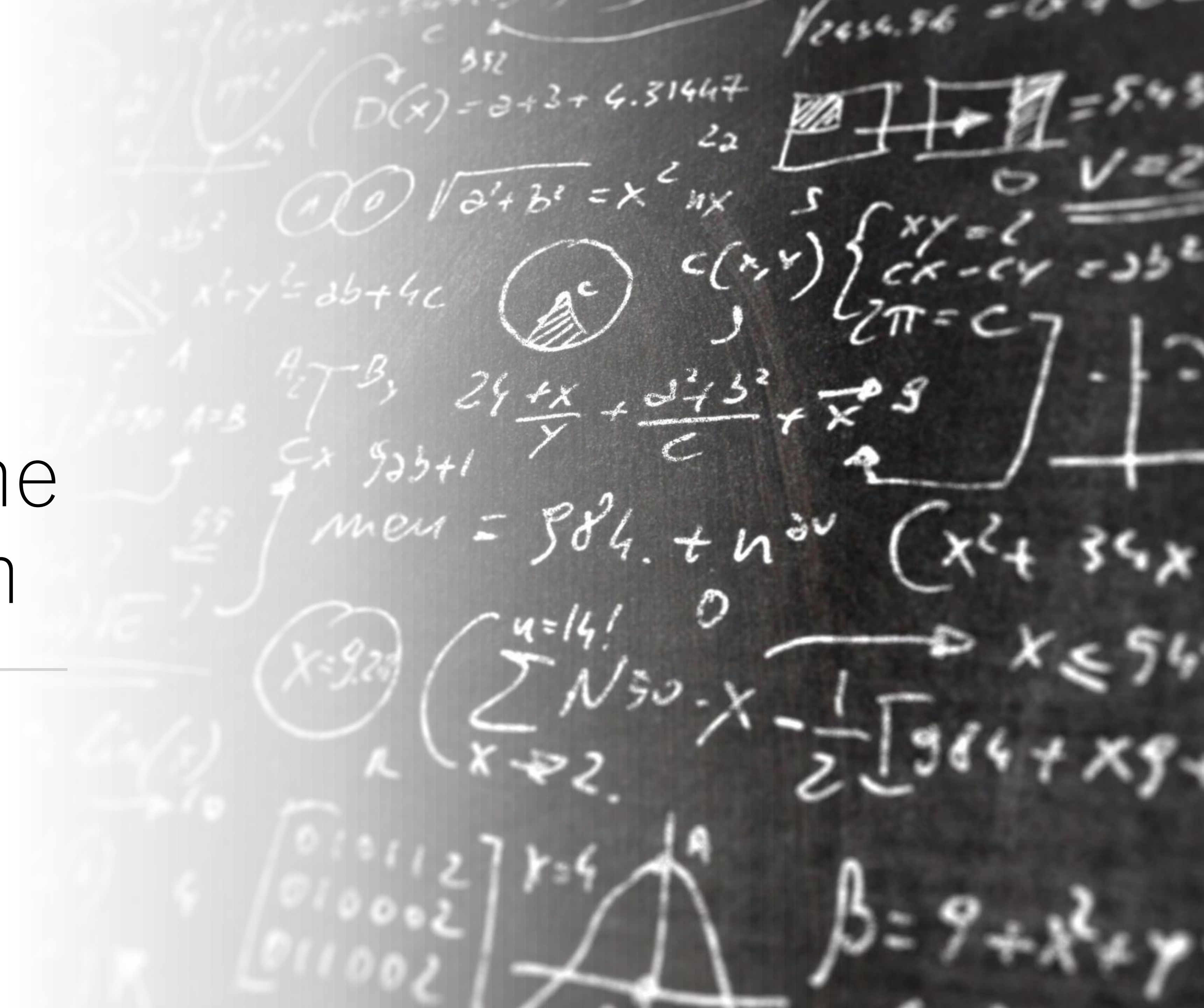
Impricit method for the heat equation



Backward difference in time (Backward Euler) for the heat equation

$$\frac{\partial u}{\partial t}(x,t) = c\,\frac{\partial^2 u}{\partial x^2}(x,t) \qquad\qquad in \quad (a,b)\times(0,T)$$

Discretize both sides in the following way:

$$\frac{\partial u}{\partial t}(x,t) \approx \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t},$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}$$

- Then we get

$$\frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}.$$

$$u(x,t) - \left(\frac{c\Delta t}{(\Delta x)^2}\right)\left(u(x, +\Delta\ t) + 2u(x,t) + u(x - \Delta x, t)\right)$$
$$= u(x, t - \Delta t)$$

- If we write $u(i\Delta x, j\Delta\ t) = u_{i,j}$ we have

$$u(x,t) - \left(\frac{c\Delta t}{(\Delta x)^2}\right)\left(u(x, +\Delta\, t) + 2u(x,t) + u(x - \Delta x, t)\right)$$
$$= u(x, t - \Delta t)$$

- If we write $u(i\Delta x, j\Delta\, t) = u_{i,j}$ we have

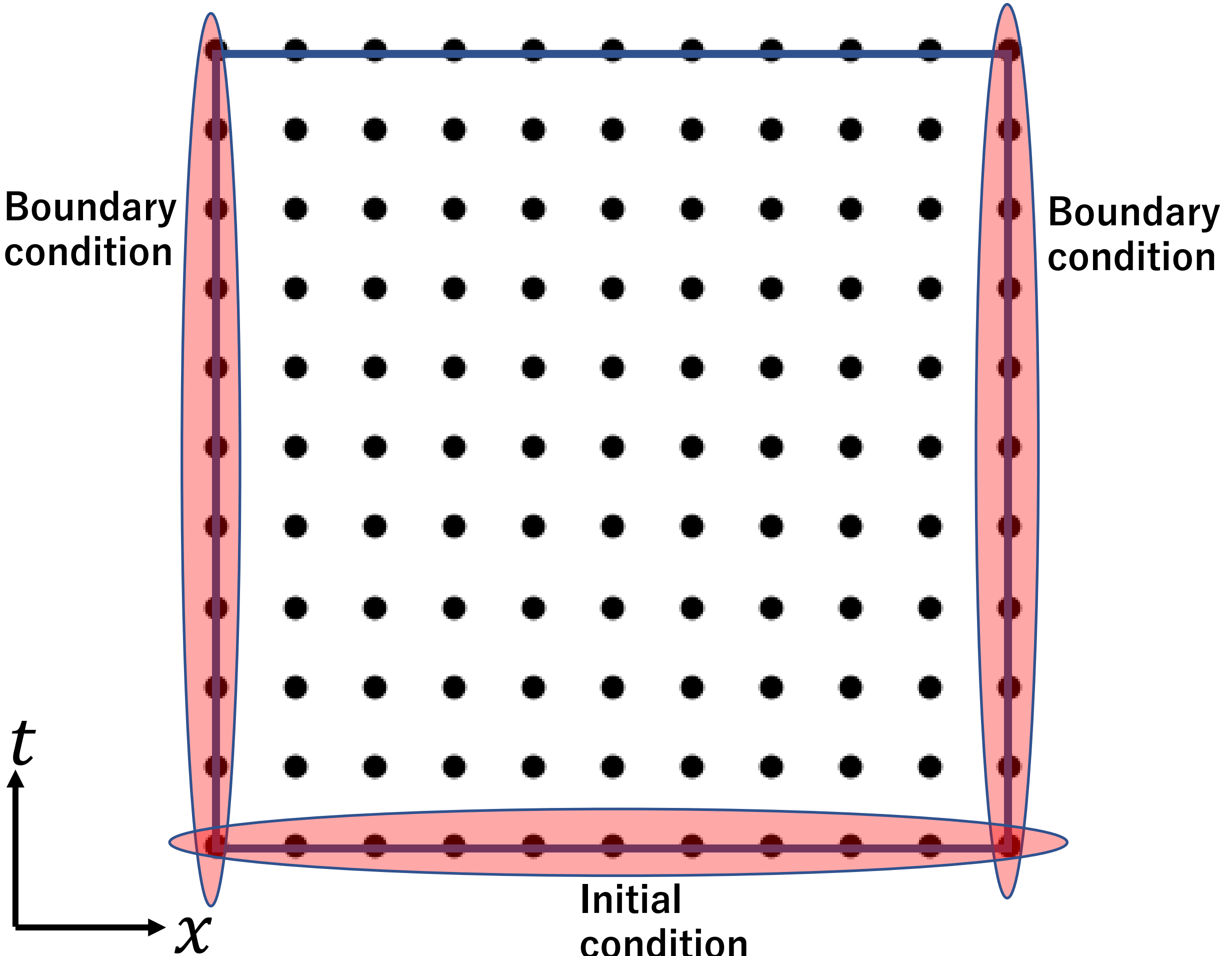
- To do this, discretize space and time.

space: $a, a+\Delta x, a+2\Delta x, \cdots, a+M\Delta x = b, M := \frac{b-a}{\Delta \mathrm{x}}$

time: $0, \Delta t, 2\Delta t, \cdots, N\Delta t = T, N := \frac{T}{\Delta t}$

By the appropriate choice

$(i\Delta x, j\Delta t)$ fit the grids



$$\left(1-2\left(\frac{c\Delta t}{(\Delta x)^2}\right)\right)u_{i,j}-\left(\frac{c\Delta t}{(\Delta x)^2}\right)(u_{i+1,j}+u_{i-1,j})=u_{i,j-1}$$

Solve $u_{i,j}\ \ (i=1,\cdots,M-1)$ using $u_{i,j-1}\ \ (i=0,\cdots,M)$.

And above equation can be translated into matrixial equation.

$$\begin{pmatrix} 1+2a & -a & 0 & \cdots & \cdots & 0 \\ -a & 1+2a & -a & 0 & \cdots & 0 \\ 0 & -a & 1+2a & -a & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -a & 1+2a & -a \\ 0 & \cdots & 0 & 0 & -a & 1+2a \end{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ \vdots \\ \vdots \\ u_{N-2,j} \\ u_{N-1,j} \end{pmatrix} = \begin{pmatrix} u_{1,j-1} + a u_{0,j} \\ u_{2,j-1} \\ \vdots \\ \vdots \\ \vdots \\ u_{N-2,j-1} \\ u_{N-1,j-1} + b u_{N,j} \end{pmatrix}$$