0 Fourier Series (フーリエ級数)

Let f be a periodic function with period 2π. We shall consider the problem whether is it possible to expand f using trigonometric function in the following way:

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{0.1}$$

If it is possible, by integration each term, the coefficients, a_0, a_n, b_n, are calculated in the following form:

$$\begin{cases} a_0 &= \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\,dx, \\ a_n &= \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos nx\,dx, \\ b_n &= \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin nx\,dx. \end{cases} \tag{0.2}$$

We call the right hand side of (0.1) is Fourier expansion of f and the series is called Fourier series. In this article, we want to justify the meaning of '$\sim$'. For example, if f is continuous, the meaning of $\sim$ should be equality in the sense of continuous function. Usually, continuity of f is too strong to be assumed. So, we will show what is the appropriate meaning of '$\sim$'.

1 Preparation

In this section, we will prepare several important theorems. We will focus on Jordan-Lebesgue's theorem (in subsection 2) and introduce several lemmas.

The next theorem is very useful to estimate from above and below in some delicate cases.

Theorem 1.1 (Abel's Inequality). Let us cosider two sequence of numbers, $\{u_m\}_{m=0}^n$ and $\{v_m\}_{m=0}^n$, and $\{u_m\}$ is assumed to be nonnegative and nonincreasing sequence, i.e., $u_0 \geq u_1 \geq \cdots u_{n-1} \geq u_n \geq 0$. Define $\sigma_p := \sum_{m=0}^{p} v_m$ $(0 \leq p \leq n)$, then summation

$$S := u_0v_0 + u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

satisfies the following inequalities:

$$\Big(\min_{0\leq p\leq n} \sigma_p\Big)u_0 \leq S \leq \Big(\max_{0\leq p\leq n} \sigma_p\Big)u_0.$$

Proof. S can be written in the following way:

$$\begin{aligned} S &= u_0\sigma_0 + u_1(\sigma_1 - \sigma_0) + +u_1(\sigma_2 - \sigma_1) + \cdots + u_n(\sigma_n - \sigma_{n-1}) \\ &= \sigma_0(u_0 - u_1) + \sigma_1(u_1 - u_2) + \cdots + \sigma_n u_n. \end{aligned}$$

Because each $u_{j-1} - u_j \geq 0$ and $u_n \geq 0$, we have

$$S = \sigma_0(u_0 - u_1) + \sigma_1(u_1 - u_2) + \cdots + \sigma_n u_n \leq \Big(\max_{0\leq p\leq n} \sigma_p\Big) \sum_{j=1}^{n}(u_{j-1} - u_j).$$

We get the estimate from above. From below, we can do the same way. □

Theorem 1.2 (Bonnet's Theorem : ボンネの定理). Let f be a Reimann integrable function defined on finite interval $[a, b]$, and φ be a nonnegative and nonincreasing function. Then, there exists $\xi \in [a, b]$ such that

$$\int_a^b \varphi(x)f(x)\,dx = \varphi(a + 0)\int_a^{\xi} f(x)\,dx.$$

Proof. Without loss of generality, we can assume $\varphi(a) = \varphi(a + 0)$. Divide $[a, b]$ into n equal subintervals and their nodal points are

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Put

$$h = x_i - x_{i-1} = \frac{b - a}{n}.$$

By Abel's inequality, (we apply with $u_i = f(x_i)$ and $v_i = \varphi(x_i)$) there exist k, k' such that

$$\varphi(a + 0)h\sum_{i=0}^{k'} f(x_i) \leq h\sum_{i=0}^{N-1} \varphi(x_i)f(x_i) \leq \varphi(a + 0)h\sum_{i=0}^{k} f(x_i). \tag{1.1}$$

By the definition of Reimann integration, i.e. $\lim_{n\to\infty} h\sum_{i=0}^{n} f(x_i) = \int_a^b f(x)\,dx$,

For any $\varepsilon > 0$, there exits h such that

$$h\sum_{i=0}^{k'} f(x_i) \leq \max_{\xi}\int_a^{\xi} f(x)\,dx + \varepsilon$$

hold. Selecting h smaller enough, and (1.1), we get

$$\varphi(a + 0)\Big[\min_{\xi}\int_a^{\xi} f(x)\,dx - \varepsilon\Big] \leq \int_a^b \varphi(x)f(x)\,dx \leq \varphi(a + 0)\Big[\max_{\xi}\int_a^{\xi} f(x)\,dx + \varepsilon\Big].$$

By arbitral choice of $\varepsilon > 0$ and continuity of $\displaystyle\int_a^{\xi} f(x)\,dx$ with respect to ξ we get the theorem. □

Remark 1.1. If we assume φ is positive and nondecreasing function, then there exists $\xi \in [a, b]$ such that

$$\int_a^b \varphi(x)f(x)\,dx = \varphi(b)\int_{\xi}^{b} f(x)\,dx.$$

holds.

Here we introduce so called Riemann-Lebesgue theorem.

Theorem 1.3 (Riemann-Lebesgue). For $\psi(t) \in L^1([a, b])$, the following holds:

$$\int_a^b \psi(t)\sin nt\,dt \to 0 \quad \text{as } n \to \infty.$$

Note. We can generalize this in the following: Let λ be a real paremeter, if $\psi(t) \in L^1((-\infty,\infty))$, thus,

$$\int_{-\infty}^{\infty} \psi(t) \sin \lambda t \, dt \to 0 \quad \text{as } \lambda \to \infty.$$

Proof. We divide $[a,b]$ with ('sin' wave) nodal points, $\frac{k}{n}\pi$ $(k = 0, \pm 1, \pm 2, ...)$, and decomposit the integral into

$$\sum_p \Big(\int_{\frac{2p}{n}\pi}^{\frac{2p+1}{n}\pi} + \int_{\frac{2p+1}{n}\pi}^{\frac{2p+2}{n}\pi} \Big) + \int_a^{a+\delta} + \int_{b-\delta'}^{b},$$

where $a+\delta$ is the first sin's nodal point (i.e. $\frac{2p}{n}\pi$) starting from a and $b-\delta'$ is a last nodal points to b. Now, $\delta, \delta' < \frac{2p}{n}\pi$, thus δ and δ' are approaching to 0 when $n \to \infty$. Since $\psi(t)$ is summable, the latter two terms of above approaching to 0 when $n \to \infty$.

If we put $t - \frac{\pi}{n} = t'$ in the second integral of above, we have $\sin nt = \sin n\Big(t' + \frac{\pi}{n}\Big) = -\sin nt'$. Thus the sum of the first term and second term will be

$$\sum_p \int_{\frac{2p}{n}\pi}^{\frac{2p+1}{n}\pi} \Big[\psi(t) - \psi\Big(t + \frac{\pi}{n}\Big)\Big] \sin nt \, dt.$$

We can estimate from above by

$$\Big|\sum_p \int_{\frac{2p}{n}\pi}^{\frac{2p+1}{n}\pi} \Big[\psi(t) - \psi\Big(t + \frac{\pi}{n}\Big)\Big] \sin nt \, dt\Big| \le \int_a^b \Big|\psi(t) - \psi\Big(t + \frac{\pi}{n}\Big)\Big| \, dt. \tag{1.2}$$

Since $\psi(t)$ is summable, by Lebesgue's theorem,
we have

$$\int_a^b |\psi(t+h) - \psi(t)| \, dt \to 0 \quad \text{as } h \to 0,$$

and the right hand side of the inequality (1.2) tends to 0 when $n \to \infty$. □

We here just mention about Lebesgue's theorem.

Theorem 1.4 (Lebesgue). For $\psi(t) \in L^p([a,b])$, $p \ge 1$, then the following holds:

$$\int_a^b |\psi(t+h) - \psi(t)|^p \, dt \to 0 \quad \text{as } h \to 0.$$

2 Jordan-Lebesgue

Theorem 2.1 (Jordan-Lebesgue). Let $f \in L^1_{\text{loc}}(\mathbb{R})$ be a 2π-periodic function and f is bounded variation near x, and put

$$S_n(x) := \frac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)$$

then the following holds:

$$S_n(x) \to \frac{f(x+0) + f(x-0)}{2} \quad (\text{as } n \to \infty).$$

Proof. By the definition of S_n, (by changing Fourier coefficient into integral form),

$$\begin{aligned} S_n &= \frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(\theta)\,d\theta \\ &\qquad + \frac{1}{\pi}\sum_{m=1}^{n}\Big[\cos mx \int_{\alpha}^{\alpha+2\pi} f(\theta)\cos m\theta\,d\theta + \sin mx\int_{\alpha}^{\alpha+2\pi} f(\theta)\sin m\theta\,d\theta\Big] \\ &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(\theta)\Big[\frac{1}{2} + \sum_{m=1}^{n}\cos m(x-\theta)\Big]\,d\theta \quad (\because \text{ addition theorem (加法定理) }). \end{aligned}$$

Again, we use addition theorem for trigonometric function,

$$2\cos 2pt\sin t = \sin(2p+1)t - \sin(2p-1)t,$$

adding both side of above,

$$\begin{aligned} \sum_{p=1}^{m} 2\cos 2pt\sin t &= \sum_{p=1}^{m}\Big(\sin(2p+1)t - \sin(2p-1)t\Big) \\ &= \sin(2m+1)t - \sin t. \end{aligned}$$

Dividing both side by $2\sin t$, we have,

$$\frac{1}{2} + \sum_{p=1}^{m}\cos 2pt = \frac{\sin(2m+1)t}{2\sin t}. \tag{2.1}$$

Put $t = \frac{x-\theta}{2}$ in above equation, we have

$$\frac{1}{2} + \sum_{p=1}^{m}\cos p(x-\theta) = \frac{\sin(2m+1)\dfrac{x-\theta}{2}}{2\sin\dfrac{x-\theta}{2}}.$$

Thus, we have

$$S_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(\theta)\frac{\sin(2n+1)\dfrac{x-\theta}{2}}{2\sin\dfrac{x-\theta}{2}}\,d\theta.$$

Here put $\theta = x + 2t$ and change integral variable $\theta \to t$, then

$$S_n = \frac{1}{\pi}\int_{\beta}^{\beta+\pi} f(x+2t)\frac{\sin(2n+1)t}{\sin t}\,dt.$$

Since α can be chosen arbitrary, thus so is β. Put $\beta = -\frac{\pi}{2}$ and divide integral domain into $\int_{-\frac{\pi}{2}}^{0}$ and $\int_{0}^{\frac{\pi}{2}}$ and change $t \to -t$ then we have

$$S_n = \int_{0}^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t}[f(x+2t) + f(x-2t)]\,dt. \tag{2.2}$$

From (2.1),

$$\int_{0}^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{2\sin t}\,dt = \frac{\pi}{4}$$

we have

$$f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t} 2f(x)\, dt,$$

(2.2) becomes

$$\pi[S_n - f(x)] = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t}[f(x+2t) + f(x-2t) - 2f(x)]\, dt.$$

Here we put

$$\varphi(t) := \frac{t}{\sin t}[f(x+2t) + f(x-2t) - 2f(x)],$$

then

$$\pi[S_n - f(x)] = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{t}\varphi(t)\, dt.$$

Now we want to show the right hand side of above tend to 0 when $n \to \infty$. For this, we divide integral area into

$$\pi[S_n - f(x)] = \int_0^{\delta} \frac{\sin(2n+1)t}{t}\varphi(t)\, dt + \int_{\delta}^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{t}\varphi(t)\, dt. \tag{2.3}$$

In the second term, since $\frac{\varphi(t)}{t}$ is summable, by Reimann-Lebesgue's theorem, thus it goes to 0 when $n \to \infty$.

On the other hand, for the first term, $f(x)$ is bounded variation in the neighbourhood of $x = x$, both $f(x+0)$ and $f(x-0)$ exits. So, we change value

$$f(x) := \frac{f(x+0) + f(x-0)}{2}.$$

We shall show when $\delta \downarrow 0$ the first term of (2.3) becomes smaller independently of m.

From the assumption, the following facts hold.

(i) $\varphi(t)$ is a BV(bounded variation) function in $[0, \delta]$

(ii) $\lim_{t \to +0} \varphi(t) = 0$.

(Remark: Definition) $f(x)$ is a function of bounded variation is iff its tatal variation,

$$V(f) := \sup_{P \in \mathcal{P}} \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)|$$

becomes finete. Here, $\mathcal{P}$ is a set of intervals P whose nodal points are $x_0, x_1, ..., x_{n_p}$.

If we put $\varphi(0) = 0$, so $\varphi(t)$ bocomes continuous at $t = 0$. On the other hand, BV functions are docomposit into

$$\varphi(t) = p(t) - n(t)$$

where $p(t)$ and $n(t)$ are $[0, t]$ positive variation and negative variation respectively. Here, $p(t), n(t)$ are non-dicreasing functions and $\lim_{t \to +0} p(t) = \lim_{t \to +0} n(t) = 0$.

By using Bonnet's inequality, (Theorem 1.2), we have

$$J_\delta := \int_0^{\delta} \frac{\sin(2n+1)t}{t} p(t)\, dt = p(\delta)\int_{\xi}^{\delta} \frac{\sin(2n+1)t}{t}\, dt \quad (0 \le \xi \le \delta)$$

and

$$\int_\xi^\delta \frac{\sin(2n+1)t}{t}\,dt = \int_{(2n+1)\xi}^{(2n+1)\delta} \frac{\sin t}{t}\,dt.$$

Here we recall well known fact,

$$\int_0^{\to\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{2.4}$$

Since absolute value of $\dfrac{\sin x}{x}$ is not integrable, i.e. in the sense of Lebesgue, it is not integrable. We can get the value of integral by using $F(\alpha) := \displaystyle\int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx$ and its limit when $\alpha \to +0$.

By (2.4), there exists positive number A, such that for any $h > 0$,

$$\left|\int_0^h \frac{\sin t}{t}\,dt\right| \le A$$

holds. Then we have

$$|J_\delta| \le 2A \cdot p(\delta).$$

Therefore, we have

$$\begin{aligned}\left|\int_0^\delta \frac{\sin(2n+1)t}{t}\varphi(t)\,dt\right| &\le 2A[p(\delta)+n(\delta)] \\ &\equiv 2A \times \underset{0\le t\le\delta}{\text{total variation}}\,[\varphi].\end{aligned} \tag{2.5}$$

Since $\varphi(t)$ is continuous at $t = 0$, $\underset{0\le t\le\delta}{\text{total variation}}\,[\varphi]$ is approaching zero accoding to δ.

Combining above and Riemann-Lebesgue's theorem,

$$\int_\delta^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{t}\varphi(t)\,dt \to 0 \quad (n\to\infty).$$

On the other hand, we have

$$\left|\int_0^\delta \frac{\sin(2n+1)t}{t}\varphi(t)\,dt\right| \le 2A \times \underset{0\le t\le\delta}{\text{total variation}}\,[\varphi] \to 0$$

Thus we got

$$S_n \to f(x) = \frac{f(x+0)+f(x-0)}{2}.$$

We will show that the convergence is uninform with respect to x. Let $x \in [\alpha', \beta']$ be a parameter, and instead using $\varphi(t)$ we shall use

$$\varphi(t;x) := \frac{t}{\sin t}[f(x+2t)+f(x-2t)-2f(x)]$$

and above argument can be done independent of x.

3 Heat equation

In this section, we will consider meaning of solution for 1 dimensional heat equation by use of Fourier series. We consider the following heat equation:

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u \qquad (x,t) \in \Omega \times (0,\infty) \tag{3.1}$$

where $\Omega = (0, \pi)$.
Boundary conditions and initial condition are the following:

$$u(0,t) = g_1(t), \quad u(\pi, t) = g_2(t), \quad (0 \le t < +\infty), \tag{3.2}$$

$$u(x,0) = f(x), \quad x \in \Omega. \tag{3.3}$$

For the sake of simplicity, here we assume

$$g_1(t) = g_2(t) \equiv 0.$$

Moreover $f(x)$ satisfies $f(0) = f(\pi) = 0$ (so called compatibility condition) and $f \in C^0(\Omega) \cap BV(\Omega)$.

At first, we consider the expansion of f. Fot this purpose at $x \in [-\pi, 0]$, we put $f(x) = -f(-x)$. Then the domain $f(x)$ can be considered $[-\pi,\pi]$. In this interval $f(x)$ is odd function so the Fourier series constructed just by $\sin nx$, $n = 1, 2, 3 \cdots$, i.e.

$$f(x) = \sum_{n=1}^{\infty} c_n \sin nx.$$

Note that f is continuous and bounded variation thus the right hand side converges uniformly in x.

The following formula is a formal solution to the heat equation:

$$u(x, t) = \sum_{n=1}^{\infty} e^{-n^2 t} c_n \sin nx. \tag{3.4}$$

We will show (3.4) is the analitical solution.
(Sketch of the proof) At first, we will show the series in (3.4) converges pointwisely in $\Omega \times (0,\infty)$. It is easy to see

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \to 0 \quad (n \to \infty).$$

By Riemann-Lebesgue's theorem, $|c_n| < M$ $(n = 1, 2, \cdots)$, moreover, $\sum e^{-n^2 t} < +\infty$, it is easy to see the series is convergent when $t > 0$.

We will show $u(x,t)$ is C^∞ function with respect to (x,t) in $t > 0$. For this, we calculate derivative by x formally each term:

$$\sum_{n=1}^{\infty} n c_n e^{-n^2 t} \sin nx \tag{3.5}$$

and taking absolute value of each term, it can be estimated like

$$\sum e^{-n^2t} n|c_n| \le M \sum_{n=1}^{\infty} n e^{-n^2t} < +\infty.$$

Thus (3.5) converges uniformly, and then the formal derivative by each term is correct. Thus,

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} n e^{-n^2t} c_n \cos nx.$$

in the same way,

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} n^2 e^{-n^2t} c_n \sin nx.$$

Here we consider derivative by t. Choose $\delta\,(>0)$ aribitrary and fixed. At $\delta \le t < \infty$, we consider formal derivative of $u(x,t)$ by t:

$$\sum n^2 e^{-n^2t}|c_n \sin nx| \le M \sum_n n^2 e^{-\delta n^2} < +\infty$$

can be estimated from above. Thus derivative by t converges uniformly then it is correct.

Then,

$$\frac{\partial u}{\partial t} = -\sum n^2 e^{-n^2t} c_n \sin nx.$$

Therefore, $u(x,\, t)$ is a solution to the heat equation at $t > 0$. In the same way, at $t > 0$, it is C^{∞}.

Next, we consider the behavior of the solution when $t \to +0$. Actually,

$$u(x,\, t) \to f(x)$$

uniformly by x. Note that the Fourier series $\sum c_n \sin nx$ が converges uniformly and $\{e^{-n^2t}\}_{n=1,2,\dots}$ is positive decreasing. Applying Abel's inequality, for arbitral ε, there exists N such that

$$\left| \sum_{n=N+1}^{\infty} e^{-n^2t} c_n \sin nx \right| < \varepsilon, \quad (t \ge 0) \tag{3.6}$$

holds. We will consider more cafully in this treatment. Since $\{e^{-n^2t}\}_{n=1,2,\dots}$ is positive nonincreasing seaquence, by Abel's inequality (Theorem 1.1), put

$$u_n = e^{-n^2t}, \quad v_n = c_n \sin nx.$$

Applying Abel's inequality, $\displaystyle\sum_{n=N+1}^{\infty} e^{n^2t} c_n \sin nx$ can be estimated from above

$$\le \max_{N \le p < \infty} \sum_{n=N}^{p} e^{-t} c_n \sin nx.$$

On the other hand, from below

$$\ge \min_{N \le p < \infty} \sum_{n=N}^{p} e^{-t} c_n \sin nx.$$

Then there exists p_0 such that,

$$\Big|\sum_{n=N+1}^{\infty} e^{-n^2t}c_n \sin nx\Big| \le \Big|e^{-t}\sum_{n=N}^{p_0} c_n \sin nx\Big| \le \Big|\sum_{n=N}^{p_0} c_n \sin nx\Big|$$

holds. On the other hand, $c_n \sin nx$ converges uniformly, for sufficient large N,

$$\Big|\sum_{n=N+1}^{\infty} e^{-n^2t}c_n \sin nx\Big| < \varepsilon.$$

Therefore, (3.6) holds.

As a consquence,

$$|u(x,t) - f(x)| \le \left|\sum_{n=1}^{N}(e^{-n^2t} - 1)c_n \sin nx\right| + \left|\sum_{n=N+1}^{\infty} e^{-n^2t}c_n \sin nx\right| + \Big|\sum_{n=N+1}^{\infty} c_n \sin nx\Big|$$

last two terms are less than 2ε independent of t, and the first term approaches to 0 when $t \to +0$. **(QED).**

4 Orthgonal system in $L^2(\Omega)$

In this section we will consider Fourier series in wider point of view.

Let Ω be an opne set in $\mathbb{R}^N$. (It is OK if $\Omega = \mathbb{R}^N$.)

Let $L^2(\Omega)$ be a set of complex valued function defined on Ω, wiht square summable. In other word,the functions above are defined in $x = (x_1, ..., x_N) \in \Omega$ and complex valued mesurable functions with

$$\int_\Omega |f(x)|^2\, dx < \infty.$$

We call above set $L^2(\Omega)$ whose norm is

$$\|f\|_{L^2(\Omega)} := \Big(\int_\Omega |f(x)|^2\, dx\Big)^{\frac{1}{2}}.$$

If the functions f and g are coincied with each other in a.e. in Ω, we consider f and g are the same element in L^2.

It is well known $L^2(\Omega)$ is **complete** Riesz-Fischer's theorem).

すなわち, $\{f_n\}$ が $L^2(\Omega)$ の Cauchy 列:

$$\|f_n - f_m\|_{L^2(\Omega)} \to 0 \quad (n, m \to \infty)$$

ならば, $f \in L^2(\Omega)$ が存在して, $\|f_n - f\|_{L^2(\Omega)} \to 0$ $(n, m \to \infty)$ が成り立つ.

We will introduce several terminology.

Definition 4.1 (Orthonormal system). We call the function series $\{\varphi_n\}$ in $L^2(\Omega)$ is a **orthonormal system**, if it satisfies in the following two conditions:

(OS1) $\{\varphi_n\}$ are orthogonal each other, i.e.

$$\int_\Omega \varphi_n(x)\overline{\varphi_m(x)}\,dx = 0 \quad (n \neq m),$$

(OS2)

$$\int_\Omega |\varphi_n(x)|^2\,dx = 1 \quad (n = 1, 2, ...)$$

Definition 4.2 ((Complete orthonormal system)). We call an orthonormal system $\{\varphi_n\}$ in $L^2(\Omega)$ is **complete** when for all $f \in L^2(\Omega)$, the following condition holds:

$$\int_\Omega f(x)\overline{\varphi_n(x)}\,dx = 0 \quad (\forall n = 1, 2, ...) \Leftrightarrow \ f = 0 \text{ a.e. } \Omega.$$

Here we consider Fourier expansion using orthonormal system. If $\{\varphi_n\}$ is orthonormal system, we consider the following Fourier expansion of $f \in L^2(\Omega)$:

$$f(x) \sim c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x) + \cdots \tag{4.1}$$

$$c_n := \int_\Omega f(x)\overline{\varphi_n(x)}\,dx. \tag{4.2}$$

We call c_n Fourier coefficient.

Note that the right hand side of (4.1) is convergent sequence in $L^2(\Omega)$. Actually,

$$\begin{aligned}
0 \leq \int_\Omega |f(x) - \sum_{i=1}^m c_i\varphi_i(x)|^2\,dx &= \int_\Omega \Big[f(x) - \sum_{i=1}^m c_i\varphi_i(x)\Big]\Big[\overline{f(x)} - \sum_{i=1}^m \overline{c_i}\overline{\varphi_i(x)}\Big] \\
= \int_\Omega |f(x)|^2\,dx + \sum_{i=1}^m |c_i|^2 \int_\Omega |\varphi_i(x)|^2\,dx &- \sum_{i=1}^m \overline{c_i}\int_\Omega f(x)\overline{\varphi_i(x)}\,dx - \sum_{i=1}^m c_i \int_\Omega \overline{f(x)}\varphi_i(x)\,dx \\
= \int_\Omega |f(x)|^2\,dx - \sum_{i=1}^m |c_i|^2
\end{aligned}$$

therefore,

$$\int_\Omega |f(x) - \sum_{i=1}^m c_i\varphi_i(x)|^2\,dx = \int_\Omega |f(x)|^2\,dx - \sum_{i=1}^m |c_i|^2. \tag{4.3}$$

Thus we have $\sum_{i=1} |c_i|^2 < \infty$ and

$$\sum_{i=1}^\infty |c_i|^2 \leq \int_\Omega |f(x)|^2\,dx \quad \text{(Bessel's inequality)}. \tag{4.4}$$

hold. By Riesz-Fischer's theorem, we have the exsistense of

$$\varphi = \lim_{m\to\infty} \sum_{i=1}^m c_i\varphi_i \quad \text{in } L^2(\Omega). \tag{4.5}$$

(∵) If we put

$$f_m(x) := \sum_{i=1}^{m} c_i \varphi_i(x),$$

then, for $m' > m$,

$$\int_\Omega |f_m(x) - f_{m'}(x)|^2\, dx = \sum_{i=1}^{m'} |c_i|^2 \to 0 \quad (m \to \infty)$$

and we can apply Riesz-Fischer's theorem.

We will show φ, Fourier series of f coincide.

At first Fourier expansion of $\varphi \in L^2$ coincide with φ itself. Actually, since $\{\varphi_i\}$ is orthogonal system,

$$\begin{aligned}\int_\Omega \varphi(x)\overline{\varphi_n(x)}\, dx &= \int_\Omega \Big(\lim_{m\to\infty} \sum_{i=1}^{m} c_i\varphi_i(x)\Big)\overline{\varphi_n(x)}\, dx = \lim_{m\to\infty}\sum_{i=1}^{m} c_i \int_\Omega \varphi_i\overline{\varphi_n(x)}\, dx \\ &= c_n = \int_\Omega f(x)\overline{\varphi_n(x)}\, dx.\end{aligned}$$

here we exchanged summation and integration by Cauchy-Schwarz's inequality.

Therefore,

$$\int_\Omega [f(x) - \varphi(x)]\overline{\varphi_n(x)}\, dx = 0 \quad (n = 1, 2, \cdots).$$

If $\{\varphi_n(x)\}$ is complete, $f = \varphi$.

Then from (4.3),

$$\sum_{i=1}^{\infty} |c_i|^2 = \int_\Omega |f(x)|^2\, dx \quad (\text{Parseval の等式}) \tag{4.6}$$

hold.

On the other hand, if (4.6) hold, then recalling (4.3), we see $f = \varphi$.

We will also consider the case that $\{\varphi_i\}$ is not complete. In this case, there exist $f \in L^2(\Omega)$, $f \neq 0$, such that it is perpendicular to any φ_i, i.e.,

$$\int_\Omega f(x)\overline{\varphi_i(x)}\, dx = 0 \quad (i = 1, 2, \ldots)$$

hold. We see Fourier coefficients of f are all 0, f's Fourier expansion $\varphi = 0$, then $f \neq \varphi$.

We can sum up above in the following theorem.

Theorem 4.3 (**L^2 Fourier expansion**)**.** The followings are equivalent:

(i) An orthonormal system φ_n is complete in $L^2(\Omega)$.

(ii) For all $f \in L^2(\Omega)$, Parseval's inequality (4.6) holds.

(iii) For any $f \in L^2(\Omega)$, f's Fourier expansion coincide with f itself.

So far, we have considered the feature of f's Fourier expansion. Here we consider on the oposite way. Let $\{\varphi_i\}$ be an complete orthonormal system, then for any complex sequence $(c) = \{c_i\}$ with

$$\sum_{i=1}^{\infty} |c_i|^2 < \infty, \tag{4.7}$$

define f by

$$f(x) := c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x) + \cdots .$$

The right hand side is defined in L^2-limit. In this case, f's Fourier series coincide with the right hand side of above.

Here we consider (c) satisfying (4.7). We write ℓ^2 the set of (c) with norm

$$\|(c)\|_{\ell^2} := \Big(\sum_{i=1}^{\infty} |c_i|^2\Big)^{\frac{1}{2}}.$$

Then, f's Fourier expansion defines an isometric mapping from $L^2(\Omega)$ into ℓ^2 by Parseval's inequality.

At the end of this secion, we remark the feature of orthonormal system, $\{\varphi_n\}$. Let $\{\varphi_n\}$ be a complete orthonormal system, then

(1) If function series $\sum c_n\varphi$ in the open set U in Ω converges uniformly, and f and each φ_n are continuous in U, then

$$f(x) = \sum_{n=1}^{\infty} c_n\varphi_n(x), \quad (x \in U)$$

holds.

(2) Even if a function series $\sum c_n\varphi$ is not converges uniformly, the following:

$$\sum_{n=1}^{\infty} |c_n||\varphi_n(x)| < \infty$$

hold on almost every U, and if Φ satisfying

$$\sum_{n=1}^{\infty} |c_n||\varphi_n(x)| < \Phi(x), \quad \int_U \Phi(x)^2\, dx < \infty$$

exists, then

$$f(x) = \sum_{n=1}^{\infty} c_n\varphi_n(x)$$

hold almost every x in U.

Example

The trigonometric system $\{1, \cos x, \sin x, \cdots, \cos nx, \sin nx, \cdots\}$ on $(0, 2\pi)$ is complete in $L^2(0, 2\pi)$.

Let f be a 2π periodic function in C^1, then Fourier coefficients satisfy the following relation

$$\pi a_n = \int_0^{2\pi} f(x)\cos nx\, dx = -\frac{1}{n}\int_0^{2\pi} f'(x)\sin nx\, dx = -\frac{\pi b_n'}{n},$$

$$\pi b_n = \int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{n}\int_0^{2\pi} f'(x)\cos nx\, dx = \frac{\pi a_n'}{n}.$$

where, (a_n', b_n') are Fourier coefficients of f'.

Thus we have f's Fourier expansion converges uniformly. Actually,

$$\sum_{n=1}^{\infty}|a_n|+|b_n| \leq \sum_{n=1}^{\infty}\frac{1}{n}(|a_n'|+|b_n'|) \leq \Big(\sum_{n=1}^{\infty}\frac{1}{n^2}\Big)^{\frac{1}{2}}\Big(2\sum_{n=1}^{\infty}|a_n'|^2+|b_n'|^2\Big)^{\frac{1}{2}}$$

hold and the right hand side is finete by Bessel's inequality. Then the right hand side is finite and is a convergence..

Therefore, Fourier series of f converges uniformly to f itself.

5 Fourier's integration formula

5.1 Fourier expansion in bounded interval

So far, we have considered Fourier expansion of 2-π periodic function. The question arouse, for the functions which have no periodicity, can we consider Fourier expansion?

Let $f \in L^1(\mathbb{R})$ be a function without period, but consider that it has '∞' periodic function.

Let us consider Fourier expansion of periodic function defined in $(-\ell, \ell)$. It can be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi}{\ell}x + b_n \sin\frac{n\pi}{\ell}x\Big)$$

where,

$$a_0 := \frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi)\,d\xi, \quad a_n := \frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi)\cos\frac{n\pi}{\ell}\xi\,d\xi, \quad b_n := \frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi)\sin\frac{n\pi}{\ell}\xi\,d\xi.$$

The purpose of this is to extend this to ∞-perodic functions. We first, consider the behavior of expansion when $\ell \to \infty$.

Note that $a_0 \to 0$, otherwise series can not be integrable. Using addtion theorem of trigonometric function, we can calculate in the following way(we exchage summation and integration formally):

$$\begin{aligned}
\sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi}{\ell}x + b_n \sin\frac{n\pi}{\ell}x\Big) &= \sum_{n=1}^{\infty}\frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi)\cos\frac{n\pi}{\ell}(x-\xi)\,d\xi \\
&= \int_{-\ell}^{\ell} f(\xi)\sum_{n=1}^{\infty}\frac{1}{\ell}\cos\frac{n\pi}{\ell}(x-\xi)\,d\xi \\
&= \frac{1}{\pi}\int_{-\ell}^{\ell} f(\xi)\Big(\sum_{n=1}^{\infty}\Delta\nu\cos\nu_n(x-\xi)\Big)\,d\xi,
\end{aligned}$$

where, $\Delta\nu := \frac{\pi}{\ell}$, $\nu_n := \frac{n\pi}{\ell}$.

If $\ell \to \infty$ then, $\Delta\nu \to 0$ and the definition of Riemann integral (precisely it is improper integral.), we have

$$\sum_{n=1}^{\infty}\Delta\nu\cos\nu_n(x-\xi) \to \int_0^{\infty}\cos\nu(x-\xi)\,d\nu.$$

Thus,

$$a_0+\sum_{n=1}^{\infty}\Big(a_n\cos\frac{n\pi}{\ell}x+b_n\sin\frac{n\pi}{\ell}x\Big)\to\frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)\int_0^{\infty}\cos\nu(x-\xi)\,d\nu\,d\xi$$
$$=\frac{1}{\pi}\int_0^{\infty}d\nu\int_{-\infty}^{\infty}f(\xi)\cos\nu(x-\xi)\,d\xi.$$

We have formally,

$$\pi f(x)=\int_0^{\infty}d\nu\int_{-\infty}^{\infty}f(\xi)\cos\nu(x-\xi)\,d\xi. \tag{5.1}$$

By above condideration, we can define Fourier transform as a limit of the Fourier series. In the next section, we will discuss precise mathematical treatment.

5.2 Fourier's integralation formula

In this subsection, we will treat above formula precisely. As we have already shown in the previous sections,

$$\pi S_n=\int_{\alpha}^{\alpha+2\pi}\frac{\sin(2n+1)\dfrac{x-\theta}{2}}{2\sin\dfrac{x-\theta}{2}}f(\theta)\,d\theta.$$

Recall that f was 2π-periodic function. We will change this formula using, この式を書き換えてみる.

$$\sin(2n+1)\frac{x-\theta}{2}=\sin n(x-\theta)\cos\frac{x-\theta}{2}+\cos n(x-\theta)\sin\frac{x-\theta}{2},$$

we have,

$$\pi S_n=\int_{\alpha}^{\alpha+2\pi}f(\theta)\sin n(x-\theta)\cot\frac{x-\theta}{2}\frac{d\theta}{2}+\frac{1}{2}\int_{\alpha}^{\alpha+2\pi}f(\theta)\cos n(x-\theta)\,d\theta.$$

The second term of the right converges to zero when $n\to\infty$ by Riemann-Lebesgue's theorem. Therefore, (if we assume that f is bounded variation in the neighbourhood of $\theta=x$, the first term is

$$\lim_{n\to\infty}\int_{\alpha}^{\alpha+2\pi}f(\theta)\sin n(x-\theta)\cot\frac{x-\theta}{2}\frac{d\theta}{2}=\frac{\pi}{2}[f(x+0)+f(x-0)].$$

In above relation, instead $f(\theta)$, we use

$$\varphi(\theta)=\begin{cases}\dfrac{2f(\theta)}{x-\theta}\tan\dfrac{x-\theta}{2}, & \Big(|\theta-x|<\delta\Big(<\dfrac{\pi}{2}\Big)\Big)\\ 0, & \text{otherwise,}\end{cases}$$

then, φ is bounded variation near $\theta=x$ and $\varphi(x\pm0)=f(x\pm0)$, by Riemann-Lebesgue's theorem, we got

$$\lim_{n\to\infty}\int_{\alpha}^{\alpha+2\pi}\frac{f(\theta)}{x-\theta}\sin n(x-\theta)d\theta=\frac{\pi}{2}[f(x+0)+f(x-0)]. \tag{5.2}$$

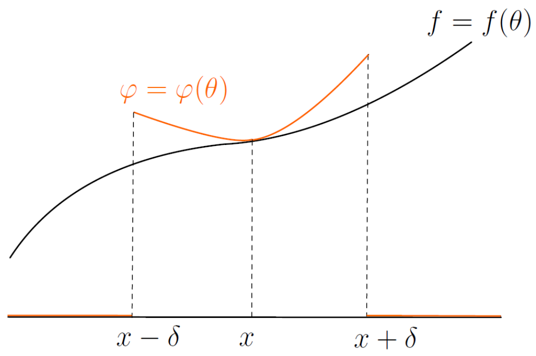

図 1: Relation between f and φ

Here we change the feature of f. So far, f was 2π-periodic function, here we ommit the assumption, and let

$$\int_{-\infty}^{+\infty} |f(\theta)|\, d\theta < \infty.$$

Again, we assume, f is bounded variation in arbitral finite interval, thus, for any x, with $\alpha < x < \alpha + 2\pi$, (5.2) hold and both

$$\int_{-\infty}^{\alpha} \frac{f(\theta)}{x-\theta} \sin n(x-\theta) d\theta, \quad \int_{\alpha+2\pi}^{\infty} \frac{f(\theta)}{x-\theta} \sin n(x-\theta) d\theta,$$

are converges to zero when $n \to \infty$ by Riemann-Lebesgue's theorem.

Conbining above facts, we have

Theorem 5.1 (Fourier's formula). Let, $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, $\forall I \subset\subset \mathbb{R}$, $f \in BV(I)$. Then このとき,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{f(\theta)}{x-\theta} \sin n(x-\theta)\, d\theta = \frac{\pi}{2}[f(x+0) + f(x-0)].$$

hold.

We will change formula in above, put

$$F_n := \int_{-\infty}^{\infty} \frac{f(\theta)}{x-\theta} \sin n(x-\theta)\, d\theta$$

and using

$$\frac{\sin n(x-\theta)}{x-\theta} = \int_0^n cos\nu(x-\theta)\, d\nu$$

we have

$$F_n = \int_{-\infty}^{\infty} d\theta \int_0^n f(\theta) \cos \nu(x-\theta)\, d\nu.$$

Here we use Fubini's theorem, we got

$$F_n = \int_0^n d\nu \int_{-\infty}^{\infty} f(\theta) \cos \nu(x-\theta)\, d\theta.$$

By above we have the following:

Theorem 5.2 (Fourier's integralatin formula). Assume as above,

$$\lim_{A\to\infty}\int_0^A d\nu\int_{-\infty}^{\infty} f(\theta)\cos\nu(x-\theta)\,d\theta = \frac{\pi}{2}[f(x+0)+f(x-0)]. \tag{5.3}$$

holds.

Note that this is same as (5.1). Moreover (5.3) is the same as Fourier's inverse transfom.

6 Fourier transform

In this section, we will define Fourier transform and discuss its feature.

For the Fourier transform, we already got its invers transformation by (5.3).

We will introduce complex valued expression, since it is used in the standard formula.

Up to now, f was assumed to be real fucntions, but almost all arguments holds for complex valude functions. Acutually, just divide into

$$f = f_1 + if_2$$

real and imaginary part. If $|f|$ is ummable, automatically f_1, f_2 are summable, and oppsit is also holds. BV is also the same.

From now on, let f be a complex valued function. For the sake of simplicity, in above (5.3), substitute ν into $2\pi\nu$, by addition formula of trigonometric functions,, we have

$$\lim_{A\to\infty} 2\int_0^A [C(\nu)\cos 2\pi\nu x + S(\nu)\sin 2\pi\nu x]\,d\nu = \frac{1}{2}[f(x+0)+f(x-0)], \tag{6.1}$$

where,

$$C(\nu) = \int_{-\infty}^{\infty} f(\theta)\cos 2\pi\nu\theta\,d\theta, \quad S(\nu) = \int_{-\infty}^{\infty} f(\theta)\sin 2\pi\nu\theta\,d\theta.$$

The equality (6.1) is the same as subustitution of integer n in $\cos 2\pi nx, \sin 2\pi nx$ in Fourier series into real parameter.

We assumed $\nu \geq 0$, but actually it has meaning when $\nu < 0$, we have

$$C(-\nu) = C(\nu), \quad S(-\nu) = -S(\nu).$$

Taking care of above, (6.1) can be written in the following:

$$\frac{1}{2}(f(x+0)+f(x-0)) = \lim_{A\to\infty}\int_{-A}^{A}[C(\nu)\cos 2\pi\nu x + S(\nu)\sin 2\pi\nu x]\,d\nu. \tag{6.2}$$

We define for f, $\hat{f}$,

$$\hat{f}(\nu) := \int_{-\infty}^{\infty} f(\theta)e^{-2\pi i\nu\theta}\,d\theta.$$

From Euler formula,

$$\hat{f}(\nu) = C(\nu) - iS(\nu).$$

therefore, in (6.2), C, S are even, odd fucntions respectively, we have,

$$\lim_{A\to\infty}\int_{-A}^{A}[C(\nu)-iS(\nu)]e^{2\pi i\nu x}\,d\nu = \lim_{A\to\infty}\int_{-A}^{A}\hat{f}(\nu)e^{2\pi i\nu x}\,d\nu.$$

Changing integral valuable, we get the following:.

Theorem 6.1 (Fourier transform and inverse transform). Let $f \in L^1(\mathbb{R})$ be BV in arbitral bounded interval. Define

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)\, dx \tag{6.3}$$

(Fourier transform), we have

$$\lim_{A\to\infty} \int_{-A}^{A} e^{2\pi i x\xi} \hat{f}(\xi)\, d\xi = \frac{1}{2}[f(x+0) + f(x-0)] \tag{6.4}$$

holds. Especially, if $\hat{f}$ is summable and f is continuous, then (6.4), we can express

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x\xi} \hat{f}(\xi)\, d\xi \tag{6.5}$$

and it is called inverse transform.

Here we will introduce the definition of Fourier transform.

Definition 6.2 (Fourier transform). Let $f \in L^1(\mathbb{R})$ be BV in any finite bounded domain. We call

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)\, dx$$

$\hat{f}$ is **Fourier transform** of f. Also, we call it Fourier image and write $\mathcal{F}f$. Here, $\mathcal{F}f = \hat{f}$.

Equalities (6.4) and (6.5) are called **Inverse formula** of Fourier transform.

The formula (6.4) is a little bit complicated, it comes from that $\hat{f}$ not ususally summable. The following is its example.

Example 1. Consider function f, for $A \geq 0$,

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq A \\ 0 & \text{if } |x| > A. \end{cases}$$

Then f's Fourier transform $\hat{f}$ is

$$\hat{f}(\xi) = \int_{-A}^{A} e^{-2\pi i x\xi}\, dx = \Big[\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\Big]_{x=-A}^{x=A} = \frac{\sin 2\pi A\xi}{\pi\xi}.$$

This $\hat{f}$ is not summable.

We will introduce several expamples for Fourier transform.

Example 2. Here, we calculate Fourier transform $f(x) = \exp\Big(-\frac{|x|^2}{2}\Big)$. By the definition,

$$\begin{aligned}\widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-2\pi i\xi x} f(x)\, dx = \int_{-\infty}^{\infty} e^{-2\pi i\xi x - \frac{|x|^2}{2}}\, dx \\ &= \int_{-\infty}^{\infty} e^{-\frac{(x+2\pi i\xi)^2}{2} - 2\pi^2\xi^2}\, dx = e^{-2\pi^2\xi^2} \int_{-\infty}^{\infty} e^{-\frac{(x+2\pi i\xi)^2}{2}}\, dx.\end{aligned}$$

The integration $\int_{-\infty}^{\infty} e^{-(x+2\pi i\xi)^2/2}\, dx$ can be calculated the following manner.

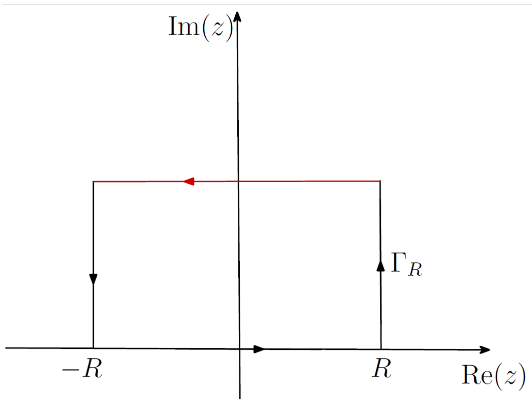

図 2: Integrate along Γ_R, Red line is $\{z = x + 2\pi i\xi \;:\; -R \leq x \leq R\}$

We introduce complex function $e^{-z^2/2}$. The above integration can be calculated using complex intergral. As in the graph of 2, integralte along red line, $\{z = x + 2\pi i\xi \;:\; -R \leq x \leq R\}$ and taking limit $R \to \infty$.

Since $e^{-z^2/2}$ is a analitic function on $\mathbb{C}$, by the Cauchy's integration theorem, we have

$$\int_{\Gamma_R} e^{-z^2/2}\, dz = 0.$$

On the other hand, the right hand side above integration will be divided into four lines, and we have

$$\int_{-R}^{R} e^{-x^2/2}\, dx + \int_{0}^{2\pi\xi} e^{-(R+iy)^2/2}\, dy + \int_{R}^{-R} e^{-(x+2\pi i\xi)^2/2}\, dx + \int_{2\pi\xi}^{0} e^{-(-R+iy)^2/2}\, dy.$$

When $R \to \infty$, $|e^{-(\pm R+iy)^2/2}| \leq e^{(-R^2+y^2)/2} \to 0$ then, the second and forth term of the integration converges to 0. Then we got,

$$\int_{-\infty}^{\infty} e^{-(x+2\pi i\xi)^2/2}\, dx = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

By above arguement, we got

$$\widehat{f}(\xi) = \sqrt{2\pi} e^{-2\pi^2\xi^2}.$$

Example 3. Let $A \geq 0$, then

$$f(x) = \begin{cases} 1 \quad \text{if } 0 < x < A \\ -1 \quad \text{if } -A < x \leq 0 \\ 0 \quad \text{if } |x| \geq A. \end{cases}$$

Calculatef f's Fourier transform:

$$\hat{f}(\xi) = \int_{-A}^{0} -e^{-2\pi i x\xi}\, dx + \int_{0}^{A} e^{-2\pi i x\xi}\, dx = -\Big[\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\Big]_{x=-A}^{x=0} + \Big[\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\Big]_{x=0}^{x=A}$$
$$= \frac{1}{\pi i\xi} - \frac{e^{2\pi i A\xi} + e^{-2\pi i A\xi}}{2\pi i\xi} = \frac{1}{\pi i\xi}(1 - \cos 2\pi A\xi).$$

6.1 Feature of Fourier transform

Fact(a) f, f' are both continuous and summable, Calculating integration by part of the definition of Fourier transform, and considering $f(x) \to 0$ when $x \to \pm\infty$, we have

$$2\pi i\xi \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f'(x)\, dx, \tag{6.6}$$

moreover,

$$2\pi|\xi||\hat{f}(\xi)| \le \|f'\|_{L^1}$$

holds.

In general, if f is C^m and $f, f', \ldots, f^{(m)}$ are all summable,

$$\mathcal{F}[f^{(m)}] = (2\pi i\xi)^m \mathcal{F}[f], \tag{6.7}$$

$$|2\pi i\xi|^m \hat{f}(\xi) \le \|f^{(m)}\|_{L^1} \tag{6.8}$$

hold.

Roughly speaking, if the function and its derivative is summable, the regularity of f is proprotionate to the decay order of $\hat{f}$ at infinity.

Fact(b) Let $|x||f|$ be summable,then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)\, dx$$

it is possible to taking derivative under integration, we have

$$\hat{f}'(\xi) = \int_{-\infty}^{\infty} -2\pi i x f(x) e^{-2\pi i x\xi}\, dx.$$

More generalm f and $x^m f$ $m = 1, 2, 3, \cdots$ are summable then $\hat{f}$ is C^m functions and

$$\frac{d^m}{d\xi^m}\hat{f}(\xi) = \mathcal{F}[(-2\pi i x\xi)^m f], \tag{6.9}$$

$$|\hat{f}^{(m)}(\xi)| \le \|(2\pi x)^m f\|_{L^1} \tag{6.10}$$

hold.

This shows that if f's decay order is faster at infinity, then existense of the higher order derivative of $\hat{f}$ is guaranteed,

From the fact (a), f, f', f'' are continuous and summable then $\hat{f}$ becomes summable, the inverse transform holds in the sense of (6.5).

Fact(c) Equality $\mathcal{F}[f(x-h)] = e^{-2\pi i h\xi}\hat{f}(\xi)$ hold.

Actually,

$$\int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x-h)\, dx = \int_{-\infty}^{\infty} e^{-2\pi i(x+h)\xi} f(x)\, dx = e^{-2\pi i h\xi}\hat{f}(\xi),$$

hold.

Here we discuss important equality about Foureir series and Parseval's equality.

This is a special case of so called Plancherel's theorem.

Lemma 6.1. Let f be C^2 function and f, f', f'' are all summable, then the following equality holds:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi. \tag{6.11}$$

Proof.

As we have discussed above, from assumption, $\hat{f}$ is summable and (6.4) hold.

Therefore

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} f(x)\overline{f(x)} \, dx = \int_{-\infty}^{\infty} f(x) \, dx \int_{-\infty}^{\infty} \overline{\hat{f}(\xi) e^{2\pi i x \xi}} \, d\xi.$$

Since $f, \hat{f} \in L^1$, by Fubini's theorem,

$$\begin{aligned} &= \int_{-\infty}^{\infty} \overline{\hat{f}(\xi)} \, d\xi \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx \\ &= \int_{-\infty}^{\infty} \overline{\hat{f}(\xi)} \hat{f}(\xi) \, d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi \end{aligned}$$

hold. □

If one just see the results of above lemma, it seems OK, if we assume only $f \in L^2$. But if the assumption is just $f \in L^2$, f is not always summable, (Because the space is hole R, we can not determine inclusion relation.), the Fourier transform $\hat{f}$ can not be defined by integration.

On the other hand, from the feature of Fourier transform, $\mathcal{F}$ studied above sections, it is an isometric operator defined in dense subset of L^2.

We can extend it to the isometric operator defined on whole L^2.

Here we ommit the precise discussion. (This will be one of further work.)

Here we summarize the discussion in this section.

Theorem 6.3. Let $f = f(x)$ and $g = g(\xi)$ are summable, then

$$\begin{aligned} \mathcal{F} f &:= \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x) \, dx, \\ \overline{\mathcal{F}} g &:= \int_{-\infty}^{\infty} e^{2\pi i x \xi} g(\xi) \, d\xi, \end{aligned}$$

and we call them Fourier transform and Inverse Fourier transform respectively.

Moreover, if f, g are C^2 and up to second derivative, they are summable, then

$$\overline{\mathcal{F}} \mathcal{F} f = f \tag{6.12}$$

$$\mathcal{F} \overline{\mathcal{F}} g = g \tag{6.13}$$

hold and

$$\|\mathcal{F} f\|_{L^2} = \|f\|_{L^2}, \quad \|\overline{\mathcal{F}} g\|_{L^2} = \|g\|_{L^2}. \tag{6.14}$$

Proof. We just should show (6.13). If g is summable and

$$\overline{\mathcal{F}}g = \overline{\mathcal{F}\overline{g}}$$

hold and and thus we have

$$\mathcal{F}\overline{\mathcal{F}}g = \mathcal{F}\cdot\overline{\mathcal{F}\overline{g}} = \overline{\overline{\mathcal{F}}\mathcal{F}\overline{g}}.$$

Applying (6.12,), we got

$$= \overline{\overline{g}} = g.$$

□

Appendix 1. Heat equation and Fourier transform

In this section, we consider how to solve the heat equation in R^1.

$$\begin{cases} u_t = u_{xx} & \text{in } (x,t) \in (-\infty,\infty)\times(0,\infty) \\ u(x,0) = f(x) \end{cases}$$

Apply Fourier transform to the both side of above equation, then

$$\begin{aligned} (\,\text{left}\,) &= \widehat{u}_t = \frac{d}{dt}\widehat{u} \\ (\,\text{right}\,) &= \int_{-\infty}^{\infty} e^{-2\pi i\xi x} u_{xx}\,dx \\ &= (2\pi i\xi)^2 \int_{-\infty}^{\infty} e^{-2\pi i\xi x} u\,dx \\ &= (2\pi i\xi)^2 \widehat{u}. \end{aligned}$$

If we write $\widehat{u}(\xi,t)$ as a transformed function, then the heat equation will be

$$\begin{cases} \dfrac{d}{dt}\widehat{u} = (2\pi i\xi)^2\widehat{u} \\ \widehat{u}(\xi,0) = \widehat{f}(\xi). \end{cases}$$

This became ordinary differential equation of $\widehat{u}$. For every ξ, this is ture. We solve this equation, then

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{(2\pi i\xi)^2 t} = \widehat{f}(\xi)e^{-4\pi^2\xi^2 t}.$$

Apply inverse Fourier transform to above and allow to exchange integration order, then,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} e^{2\pi i\xi x}e^{-4\pi^2\xi^2 t}\Big(\int_{-\infty}^{\infty} e^{-2\pi i y\xi} f(y)\,dy\Big)\,d\xi \\ &= \int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} e^{-2\pi i(y-x)\xi}e^{-4\pi^2\xi^2 t}\,d\xi\,dy \\ &= \int_{-\infty}^{\infty} f(y)\widehat{e^{-4\pi^2\xi^2 t}}(y-x)\,dy. \end{aligned}$$

Recall,

$$\widehat{e^{-4\pi^2\xi^2 t}}(z) = \frac{1}{\sqrt{4\pi t}}\exp\Big(-\frac{z^2}{4t}\Big) \tag{A.1}$$

then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp\Bigl(-\frac{(x-y)^2}{4t}\Bigr) f(y)\, dy.$$

Thus we got a solution.

Here, we prove the equation (A.1). For this purpose, apply the following theorem with changing $a = 4\pi t$.

Theorem. Fourier transform of $f(x) = e^{-\pi a x^2}$ is

$$\widehat{f}(\xi) = a^{-\frac{1}{2}} \exp\Bigl(-\frac{\pi \xi^2}{a}\Bigr).$$

Proof. By using change of variable, $x \mapsto x/\sqrt{a}$, we should show just the case $a = 1$. Thus we calculate

$$\begin{aligned}\widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-2\pi i \xi x} e^{-\pi x^2}\, dx \\ &= \int_{-\infty}^{\infty} e^{-\pi (x+i\xi)^2} e^{-\pi \xi^2}\, dx \\ &= e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi (x+i\xi)^2}\, dx.\end{aligned}$$

Now, integration of $e^{-\pi z^2}$, if $|\mathrm{Im} z|$ bounded, by the Cauchy's integration theorem, the line integral can be moved on $\mathrm{Im} z = 0$. Then,

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$$

thus we have

$$\widehat{f}(\xi) = e^{-\pi \xi^2}.$$

□

Appendix 2. The fundamental solution

Here we consider the following partial differental equation.

$$\frac{\partial E}{\partial t} - E_{xx} = \delta(x,t) \quad (x,t) \in \mathbb{R} \times \mathbb{R} \tag{A.2}$$

Here, $\delta(x,t) = \delta(x)\delta(t)$. The meaning of above, we consider

$$\int_{-\infty}^{\infty} \delta(x)\delta(t)\varphi(t)\, dt = \delta(x) \int_{-\infty}^{\infty} \delta(t)\varphi(t)\, dt,$$

something like above. Here, we racall, the distribution sense of derivative of Heaviside function will be the Dirac's delta function, δ. The Heaviside function is defined

$$H(t) = \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0). \end{cases}$$

Note that the definition is sometimes different from books and papers, but the important thing is that the origin it is discontinuous dispite the value of it. For any $\varphi \in C_c^1(\mathbb{R})$,

$$\begin{aligned}\int_{-\infty}^{\infty} H'(t)\varphi(t)\,dt &= -\int_{-\infty}^{\infty} H(t)\varphi'(t)\,dt \quad (\because \text{ definiton of derivative})\\ &= -\int_{0}^{\infty} \varphi'(t)\,dt \quad (\because \text{ definition of Heaviside})\\ &= \varphi(0) = \int_{-\infty}^{\infty} \delta(t)\varphi(t)\,dt \quad (\because \text{ definition of delta})\end{aligned}$$

Apply Fourier transform to (A.2) with respect to x, we have

$$\frac{d}{dt}\widehat{E}(\xi,t) + 4\pi^2\xi^2\widehat{E}(\xi,t) = \delta(t). \tag{A.3}$$

Here, we used $\widehat{\delta(x)} = 1$. Then (A.3) can be considered to be the following ordinary differential equation:

$$\frac{d}{dt}F(t) - \alpha F(t) = \delta(t).$$

Put $F(t) = e^{at}G(t)$, then

$$\frac{dG}{dt} = e^{-at}\delta(t).$$

Thus, for any $\varphi \in C_0^\infty(-\infty,\infty)$, we see from

$$\langle e^{-at}\delta(t), \varphi(t)\rangle := \int_{-\infty}^{\infty} e^{-at}\delta(t)\varphi(t)\,dt = e^{-a\cdot 0}\varphi(0) = \varphi(0),$$

$e^{-at}\delta(t)$ is coincide with $\delta(t)$. Then the equation is

$$\frac{dG}{dt} = \delta(t).$$

This means

$$G(t) = H(t) + C,$$

where H is the Heaviside function and C is a constant. Now, we choose initial data to be $C = 0$, we see $F(t) = e^{at}H(t)$. From this we have

$$\widehat{E}(\xi,t) = H(t)e^{-4\pi\xi^2 t}.$$

Apply Fourier inverse transform, we have 逆変換により,

$$\begin{aligned}E(x,t) &= H(t)(4\pi t)^{-\frac{1}{2}}\exp\Big(-\frac{x^2}{4t}\Big)\\ &= H(t)K_t(x).\end{aligned}$$

We call this is the fundamental solution to the heat equation.
Note.

$$\begin{aligned}u(x,t) &= g * E \quad (\text{convolution})\\ &= \int_{-\infty}^{t}\int_{\mathbb{R}} g(y,s)E(x-y,t-s)\,dy\,ds\end{aligned}$$

is a formal solution to

$$\frac{\partial u}{\partial t} = \Delta u + g(x, t).$$

Proof

$$\begin{aligned}(\partial_t - \Delta)(g * E) &= g * (\partial_t - \Delta)E \\ &= g * \delta \\ &= g\end{aligned}$$

本稿について

本稿は Fourier 解析を主題とした 2021 年度解析学 3C(金沢大学理工学域) の講義ノートである. 作成するにあたり, 以下の書籍を参考にしている.

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