1 Discretization of 2 dimensional elliptic equation

1.1 Weak formula of Laplace equation

Here, we will consider discretization of Laplace equation by using finite element method. Strong formula of Laplace equation is the following formula:

$$\begin{aligned} \Delta u &= 0 \quad \text{on } \Omega \subset \mathbf{R}^n \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

A weak formula is

$$\begin{aligned} -\int_\Omega \nabla u \cdot \nabla \zeta dx &= 0 \quad {}^\forall \zeta \in C_0^\infty(\Omega) \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

We use this form as a starting point. We assume $\Omega \subset \mathbf{R}^2$.

1.2 Finite element decomposition

We divide $\Omega \subset \mathbf{R}^2$ into set of triangle subdomain (we call this **finite element** or simply **element**. We call vertices of each triangle **node**.) This decomposition should be

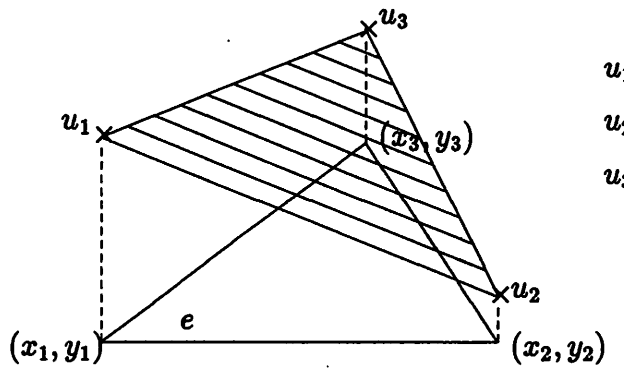
(1) Each edge (side) of triangle should be fit neighbor's edge without gap and over lapping

(2) Vertices of triangle common for each neighbor triangle

(3) For triangle on the boundary, vertices should be put on boundary point

1.3 Area coordinate

We introduce area coordinate. This is one of the transforms from arbitral triangle to rectangular equilateral triangle whose peak is on origin.



$u_1 : u(x_1, y_1)$
$u_2 : u(x_2, y_2)$
$u_3 : u(x_3, y_3)$

Let $\hat{u}(x, y)$ be a piecewise linear function which approximates $u(x, y)$ on a finite element e, then we can write

$$\hat{u}(x, y) = \alpha x + \beta y + \gamma. \tag{1.1}$$

Here α, β and γ are constants which will be defined.

Let (x_i, y_i), $(i = 1, 2, 3)$ be nodal points on each finite element e and Let u_i $(i = 1, 2, 3)$ be a value of u on each nodal point. Then, since

$$\begin{cases} u_1 = \alpha x_1 + \beta y_1 + \gamma \\ u_2 = \alpha x_2 + \beta y_2 + \gamma \\ u_3 = \alpha x_3 + \beta y_3 + \gamma \end{cases}$$

hold then we have

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Here, if we put

$$D := \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$D \neq 0$ because the nodal points not on one line then, by Cramer's rule we have

$$\alpha = \frac{\begin{vmatrix} u_1 & y_1 & 1 \\ u_2 & y_2 & 1 \\ u_3 & y_3 & 1 \end{vmatrix}}{D}, \quad \beta = \frac{\begin{vmatrix} x_1 & u_1 & 1 \\ x_2 & u_2 & 1 \\ x_3 & u_3 & 1 \end{vmatrix}}{D}, \quad \gamma = \frac{\begin{vmatrix} x_1 & y_1 & u_1 \\ x_2 & y_2 & u_2 \\ x_3 & y_3 & u_3 \end{vmatrix}}{D}.$$

For $(i, j, k) = (1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, if we define

$$a_i := \frac{y_j - y_k}{D}, \quad b_i := \frac{x_k - x_j}{D}, \quad c_i := \frac{x_j y_k - x_k y_j}{D},$$

we can write

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

and

$$\alpha = \sum_{i=1}^{3} a_i u_i, \quad \beta = \sum_{i=1}^{3} b_i u_i, \quad \gamma = \sum_{i=1}^{3} c_i u_i, \tag{1.2}$$

holds. Therefore, by (1.1), (1.2), we have

$$\hat{u}(x, y) = \sum_{i=1}^{3} (a_i x + b_i y + c_i) u_i$$

and if we put

$$\lambda_i(x, y) := a_i x + b_i y + c_i \tag{1.3}$$

then we can write

$$\hat{u}(x, y) = \sum_{i=1}^{3} \lambda_i u_i. \tag{1.4}$$

We call $(\lambda_1, \lambda_2, \lambda_3)$ **area coodinate**. By using this derivative of $\hat{u}$ by x, y can be expressed just by area coodinate, i.e.

$$\frac{\partial \hat{u}}{\partial x} = \sum_{i=1}^{3} \frac{\partial \lambda_i}{\partial x} u_i = \sum_{i=1}^{3} a_i u_i$$

$$\frac{\partial \hat{u}}{\partial y} = \sum_{i=1}^{3} \frac{\partial \lambda_i}{\partial y} u_i = \sum_{i=1}^{3} b_i u_i.$$

1.4 Basis function on rectangular domain

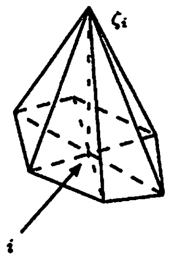
In this subsection, we will discuss basis functions for finite element. The basis function ζ_i will be defined

(1) on each nodal point i,

(2) $\zeta_i = 1$ on the node and $\zeta_i = 0$ on the edge(side) of neighbor element of nodal point,

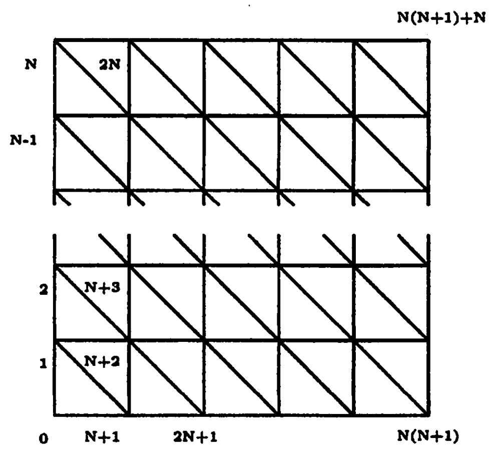
(3) Piecewise linear function on each element and identically zero outside of neighbor triangle.

The graph of it in the following form.



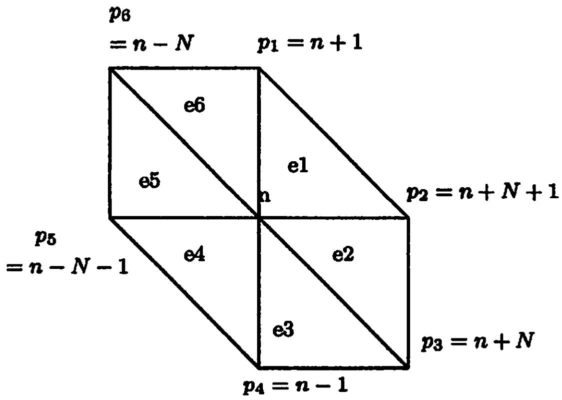
1.5 An example of integration using basis function

We forcus on square domain Ω and divided this N equal length in x axis direction and y axis also. Each small square, its edge length is equal to Δx. Put number on each node in the following way:



Nodal point at number at $(i \times \Delta x, j \times \Delta x)$ is $i(N+1)+j$.
$(i = 0, \cdots, N\ ,\ j = 0, \cdots, N)$

Here, put $n = i(N+1)+j$ and give node number for triangles for which nth node is belonging to as $p_1, p_2, \cdots, p_6$. And for each triangle domain, we give number as $e_{n_1}, e_{n_2}, \cdots, e_{n_6}$.



Let us consider the basis function ζ_n which takes value one on n th nodal point. On the element which $\zeta_n \neq 0$, discretize the following

$$-\int\int_{\Omega} \nabla u \cdot \nabla \zeta_n dxdy = 0. \tag{1.5}$$

(We do not need to consider above integral on the element which satisfies $\zeta_n \equiv 0$, because integral become zero. On the element e_{n_k}, we apply linear approximation to u by (1.4), then

$$u(x,y) = \widehat{u}^{(e_{n_k})}(x,y) = \sum_{m=n,p_\alpha,p_\beta} \lambda_m^{(e_{n_k})} u_m \qquad \text{in } (x,y) \in e_{n_j}. \tag{1.6}$$

Note that $\lambda_m^{(e_{n_j})}$ is a area coordinate defined by (1.3) the elemente_{n_j}. And u_m is value u on the m^{th}. By the definition of basis function ζ_n, we have

$$\zeta_n(x,y) = \begin{cases} \lambda_n^{(e_{n_1})} & (x,y) \in e_{n_1} \\ \lambda_n^{(e_{n_2})} & (x,y) \in e_{n_2} \\ \lambda_n^{(e_{n_3})} & (x,y) \in e_{n_3} \\ \lambda_n^{(e_{n_4})} & (x,y) \in e_{n_4} \\ \lambda_n^{(e_{n_5})} & (x,y) \in e_{n_5} \\ \lambda_n^{(e_{n_6})} & (x,y) \in e_{n_6} \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

By (1.6) and (1.7), we calculate (1.5) then

$$
\begin{aligned}
-\int\int_{\Omega} \nabla u \cdot \nabla \zeta dxdy &= -\sum_{k=1}^{6} \int\int_{e_{n_k}} \left(\nabla(\sum_l \lambda_l^{(e_{n_k})} u_l) \cdot \nabla \lambda_n^{(e_{n_k})} \right) dxdy \\
&= -\sum_{k=1}^{6} \left(\int\int_{e_{n_k}} (\sum_l \nabla \lambda_l^{(e_{n_k})} u_l) \cdot \nabla \lambda_n^{(e_{n_k})} dxdy \right) \\
&= -\sum_{k=1}^{6} \left(\sum_l u_l \int\int_{e_{n_k}} \nabla \lambda_l^{(e_{n_k})} \cdot \nabla \lambda_n^{(e_{n_j})} dxdy \right) \\
&= -\frac{|e|}{(\Delta x)^2} (8u_n - 2u_{p_i} - 2u_{p_2} - 2u_{p_4} - 2u_{p_5}) \\
&= -(4u_n - u_{n+1} - u_{n+N+1} - u_{n-1} - u_{n-N-1}) \quad (1.8)
\end{aligned}
$$

holds. Note that area of $|e|$ is $(\frac{1}{2}(\Delta x)^2)$.
We can extract nodes on the boundary because it is determined by Dirichlet condition. If the number of nodes no the Dirichlet boundary is equal to M, the number of equation defined (1.8) is $N(N+1)+1-M$.

1.6 Example of discretization of Laplace equation

As a concrete example, we discretize the following two dimensional equation,

$$
\begin{aligned}
\Delta u &= 0 \quad \text{on } \Omega = \{(x,y) | -1 \le x, y \le 1\} \\
u &= 1 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Choose $N = 4$, as we explained before, we proceed calculation,

$$
\begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix}
u_6 \\ u_7 \\ u_8 \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{16} \\ u_{17} \\ u_{18}
\end{pmatrix}
=
\begin{pmatrix}
2 \\ 1 \\ 2 \\ 1 \\ 0 \\ 1 \\ 2 \\ 1 \\ 2
\end{pmatrix}
$$

We got above. Note that if the size of matrix equation, the elements are almost zero. We call this kind of matrix **sparse matrix**. And the treatment of sparse matrix is quite different from dense matrix in the computer.