1 Variational Problem

1.1 One dimensional case

Let $\Omega := (a, b)$ be an open interval. We consider a functional I, a map from a function space to $\mathbb{R}$, defined by

$$I(u) := \int_\Omega \left(\frac{du}{dx}\right)^2 dx,$$

where $u : \Omega \to \mathbb{R}$. Our goal is to find a minimizer of I in the set

$$\mathcal{K} := \{u \in C^1(\Omega) \cap C^0(\overline{\Omega}); u(a) = \alpha, u(b) = \beta\}.\ ^\dagger$$

Definition 1.1. *A function $u \in \mathcal{K}$ is a minimizer of I on $\mathcal{K}$ if it satisfies*

$$\inf_{v \in \mathcal{K}} I(v) = I(u).$$

Remark 1.1. *Since I is nonnegative, it is bounded from below. Hence there exists infimum of I by continuity of real numbers.*

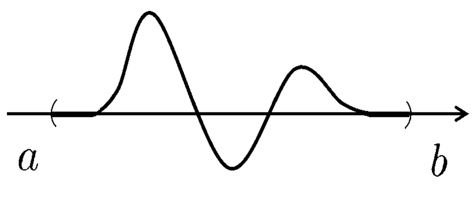
Definition 1.2. *Let $C_0^\infty(\Omega)$ be a set of smooth function whose support,*

$$\mathrm{spt}\varphi := \overline{\{x \in \mathbb{R}; \varphi(x) \neq 0\}},$$

is compact and contained in Ω.

Proposition 1.1. *Let $\varphi \in C_0^\infty(\Omega)$. Then there exists $\varepsilon_0 > 0$ such that $\varphi(x) = 0$ for any $x \in \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) < \varepsilon_0\}$.*

Example 1.1. *A graph of $\varphi \in C_0^\infty(\Omega)$ is shown in the below figure:*



Let $\varphi \in C_0^\infty(\Omega)$ and ε be a small number. We assume that u is a minimizer of I in $\mathcal{K}$. Since $C^1(\Omega) \cap C^0(\overline{\Omega})$ is a linear space and $\varphi(x) = 0$ for all $x \in \partial\Omega$, we obtain $u + \varepsilon\varphi \in \mathcal{K}$. Then

$$\begin{aligned} I(u + \varepsilon\varphi) &= \int_\Omega \left(\frac{d(u + \varepsilon\varphi)}{dx}\right)^2 dx \\ &= \int_\Omega \left(\frac{du}{dx}\right)^2 dx + 2\varepsilon \int_\Omega \frac{du}{dx}\frac{d\varphi}{dx} dx + \varepsilon^2 \int_\Omega \left(\frac{du}{dx}\right)^2 dx. \end{aligned}$$

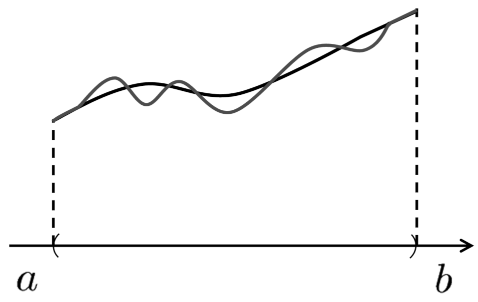


Fig. 1: Graph of u and $u+\varepsilon\varphi$

The function $\varepsilon \mapsto I(u+\varepsilon\varphi)$ must have a local minimum at $\varepsilon = 0$. Moreover, $\varepsilon \mapsto I(u+\varepsilon\varphi)$ is a differentiable function in ε. Hence we see that

$$\frac{d}{d\varepsilon} I(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = 0. \tag{1.1}$$

The left-hand side of (1.1) is called the *first variation* of I. We have by (1.1)

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} I(u+\varepsilon\varphi)\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\left(\int_\Omega \left(\frac{du}{dx}\right)^2 dx + 2\varepsilon\int_\Omega \frac{du}{dx}\frac{d\varphi}{dx}dx + \varepsilon^2\int_\Omega \left(\frac{d\varphi}{dx}\right)^2 dx\right)\Bigg|_{\varepsilon=0} \\ &= 2\int_\Omega \frac{du}{dx}\frac{d\varphi}{dx}dx. \end{aligned}$$

We rewrite the above equality as

$$-\int_\Omega \frac{du}{dx}\frac{d\varphi}{dx}dx = 0 \quad (\forall \varphi \in C_0^\infty(\Omega)). \tag{1.2}$$

Additionally, we assume that $u \in C^2(\Omega)$. From the integration by parts and (1.2), we obtain

$$\int_\Omega \frac{d^2u}{dx^2}\varphi dx = \left[\frac{du}{dx}\varphi\right]_{x=a}^{x=b} - \int_\Omega \frac{du}{dx}\frac{d\varphi}{dx}dx = 0 \quad (\forall \varphi \in C_0^\infty(\Omega)).$$

Hence we have

$$\frac{d^2u}{dx^2} = 0 \quad \text{in } \Omega. \tag{1.3}$$

We call (1.2) the *weak formulation* of (1.3). (1.3) implies that u is a linear function.

†In mathematical sense, function space should be $\{u \in W^{1,2}(\Omega); \text{with b.c.}\}$ so that a minimizer of I exists where $W^{1,2}$ is a Sobolev space with $u, u_{x_i} \in L^2(\Omega)$.

1.2 Multi dimensional case

Let $\Omega \subset \mathbb{R}^n$ be a domain and have a Lipschitz boundary. Similarly, we define a functional

$$I(u) := \int_\Omega |\nabla u|^2 dx,$$

and a function set

$$\mathcal{K} := \{u \in C^1(\Omega) \cap C^0(\overline{\Omega}); u|_{\partial\Omega} = g\},$$

where $\nabla u := \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right)$, $|\nabla u|^2 := \nabla u \cdot \nabla u$ and g is a given function. If u is a minimizer of I in $\mathcal{K}$, then u satisfies

$$-\int_\Omega \nabla u \nabla \varphi dx = 0 \quad (\forall \varphi \in C_0^\infty(\Omega)).$$

Moreover, if $u \in C^2(\Omega)$, we obtain,

$$\Delta u = 0 \quad \text{in } \Omega \quad \left(\Delta u := \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}\right). \tag{1.4}$$

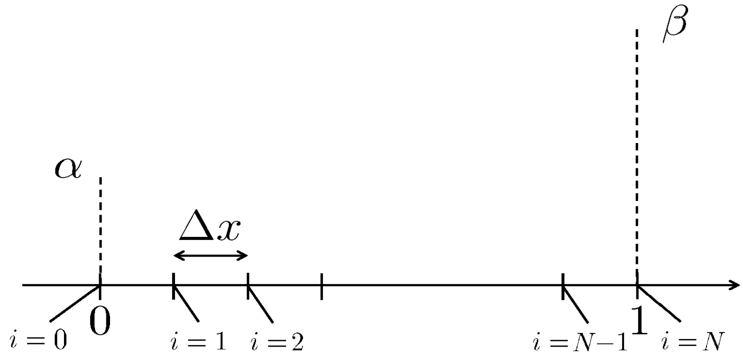
1.3 Numerical calculation

The equation (1.4) is called Laplace's equation. It is difficult for us to consider multi-dimensional Laplace's equation mathematically on arbitrary domain Ω. But the numerical calculation of this equation is not difficult.

First, let us consider it in the one-dimensional case:

$$\begin{cases} \dfrac{d^2 u}{dx^2} = 0 \quad \text{in } \Omega := (0,1) \\ u(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

Let $\Delta x := 1/N$ ($N \in \mathbb{N}$) and divide Ω into N intervals of equal length.



In this situation, we use the finite difference approximation

$$\frac{d^2u}{dx^2}(x) \simeq \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2}. \tag{1.5}$$

If $u \in C^2(\Omega)$, the quantity of (1.5) converges to $\frac{d^2u}{dx^2}$ as $\Delta x \to 0$.

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} = 0 \qquad i = 1, 2, \cdots, N-1, \tag{1.6}$$

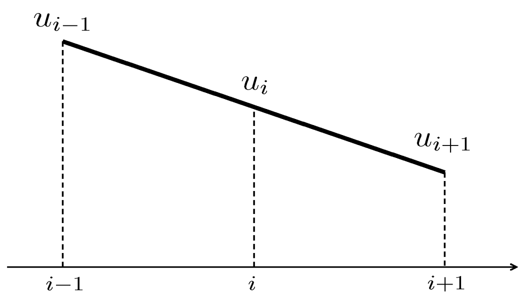
where $u_i := u(i\,\Delta x)$ $(i = 0, 1, \cdots, N)$. We restrict the equation only on the nodal points. We rewrite (1.6) as a system of linear equations of the form $Ax = b$:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}. \tag{1.7}$$

From (1.6), for each $i = 1, 2, \cdots, N-1$,

$$\begin{aligned} u_{i+1} - 2u_i + u_{i-1} &= 0, \\ u_i &= \frac{u_{i+1} + u_{i-1}}{2}. \end{aligned}$$

Hence this formula shows (is the figure part of the sentence)



For each $i = 1, 2, \cdots, N-1$, u_i is an average of u_{i+1} and u_{i-1}. Usually, a system of linear equations can be solved by using the Gaussian elimination method. This method is simple, but it may cause some troubles in a special case.

In equation (1.7), since A is a positive definite symmetric matrix, we obtain that

$$\text{Solve } Ax = b \quad \Longleftrightarrow \quad \text{Find the minimizer of } f(x) := \frac{1}{2}(x, Ax) - (b, x). \tag{1.8}$$

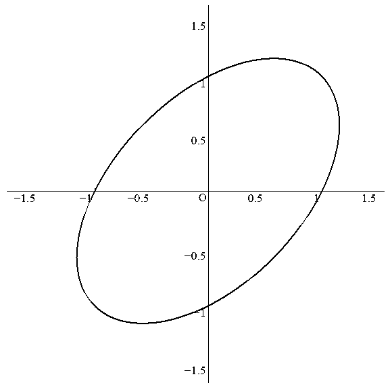
Definition 1.3. *Matrix A is positive definite if all of its eigenvalues are positive.*

Definition 1.4. *Matrix A is a symmetric matrix if it satisfies $A = A^t$ where A^t denotes the transpose of A.*

Hence, to solve (1.7), we consider a minimization problem for f . To simplify, we assume $\alpha = \beta = 0$ and $N = 3$. Then

$$f(x) = x^r \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x \qquad (x \in \mathbb{R}^2).$$

Problem. Choose $C > 0$ and plot the set $\{x; f(x) = C\}$ for $C = 1$.



The Fig.2 is the level set of f. Let $x_0 \in \mathbb{R}^{N-1}$ be an initial point of numerical computation. Firstly, calculate the gradient of $f(x)$ at x_0. It is perpendicular to the level set. Secondly, draw a line passing x_0 with direction $\nabla f(x_0)$ and find a minimum point x_1 on the line. We again calculate the gradient of $f(x)$ at x_1. We repeat this procedure. It may take a long time to calculate in this way. If A is an identity matrix, we can find the minimum in one iteration. We

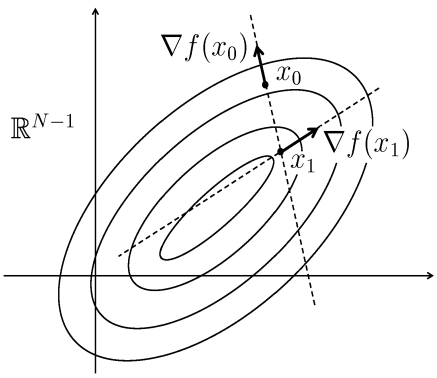


Fig. 2: Level set of f

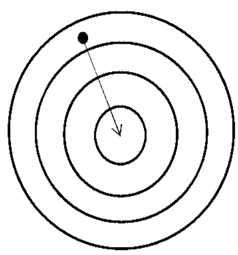


Fig. 3: Level set of f if A is an identity matrix

expect that the convergence will be fast if A is similar to an identity matrix.

Hence we find a matrix $C \sim A^{-1}$ and solve $CAx = Cb$. In the equation (1.7), we should choose

$$C = \begin{pmatrix} \frac{1}{a_{1,1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{2,2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{N-1,N-1}} \end{pmatrix}$$

But in other case, it isn't easy to find such a matrix C.

1.4 Another way to derive the system of linear equations

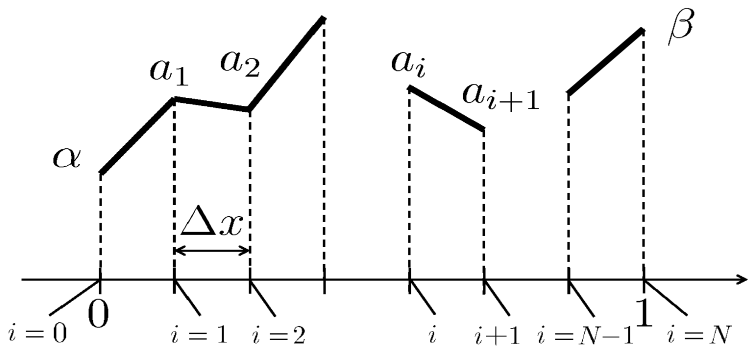
The original problem is finding the minimizer of the functional

$$I(u) = \int_\Omega \left(\frac{du}{dx}\right)^2 dx \quad (\Omega := (0,1)),$$

on the set

$$\mathcal{K} := \{v \in C^1(\Omega) \cap C^0(\overline{\Omega}); v(0) = \alpha, v(1) = \beta\},$$

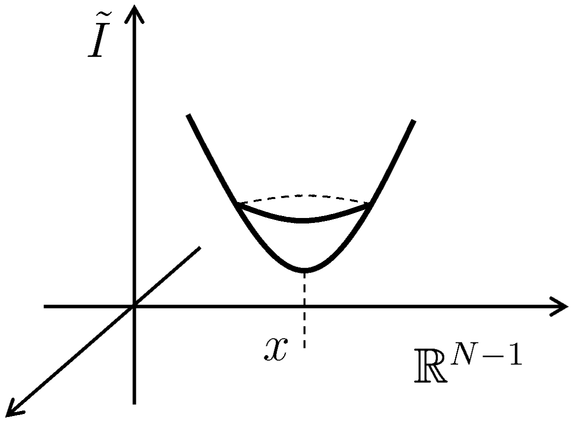
Let $\Delta x := 1/N$ ($N \in \mathbb{N}$) and divide Ω into N intervals of equal length. We approximate the function u by a piecewise linear function:



Then we can calculate I of the approximating function, and introduce functional $\tilde{I}$:

$$\tilde{I}((a_1, a_2, \cdots, a_{N-1}) = \sum_{i=1}^{N-1} \left(\frac{a_{i+1} - a_i}{\Delta x}\right)^2 \Delta x = \sum_{i=0}^{N-1} \frac{(a_{i+1} - a_i)^2}{\Delta x}. \tag{1.9}$$

$\tilde{I}$ will be the function f in (1.8) of the minimization problem.



The graph of $\tilde{I}$ is shown in the above figure. We can see that the graph is quadratic. Hence, the partial derivative of $\tilde{I}$ vanishes at a minimizer:

$$\frac{\partial \tilde{I}}{\partial a_i} = 0 \qquad i = 1, \cdots, N-1. \tag{1.10}$$

Consequently we get

$$a_{i+1} - 2a_i + a_{i-1} = 0 \qquad i = 1, \cdots, N-1. \tag{1.11}$$

We can change the above to (1.7).

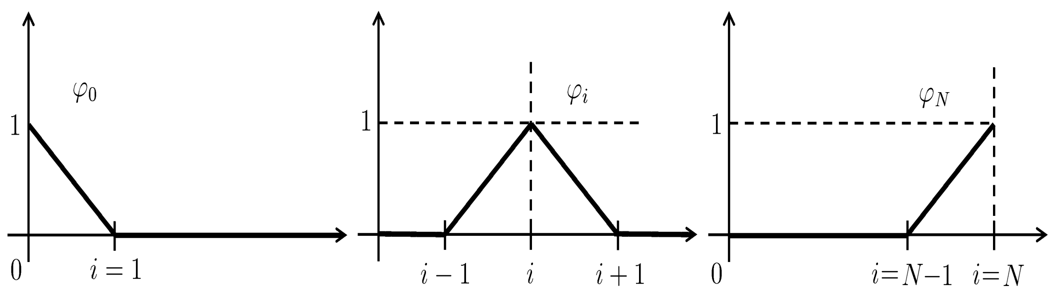
1.5 One dimensional Finite elements method

Let us consider the weak formula

$$-\int_\Omega \frac{du}{dx}\frac{d\varphi}{dx}\,dx = 0 \quad (\forall \varphi \in C_0^\infty(\Omega)). \tag{1.12}$$

We approximate u by using following base functions $\{\varphi_i\}_i^N$ as

$$u \simeq \sum_{i=0}^{N} a_i \varphi_i. \tag{1.13}$$



For each test function φ_j $(j = 1, \cdots, N-1)$, we get from (1.12)

$$-\int_\Omega \frac{d}{dx}\left(\sum_{i=0}^{N} a_i \varphi_i\right) \frac{d\varphi_j}{dx}\, dx = 0.$$

In the numerical calculation, we do not need to require that φ_j belongs to $C_0^\infty(\Omega)$. Finally, we get

$$\sum_{i=0}^{N} a_i \int_\Omega \varphi_i' \varphi_j' \, dx = 0 \qquad j = 1, \cdots, N-1.$$

Forethermore, by using the formula

$$\int_\Omega \varphi_i' \varphi_j' \, dx = \begin{cases} \frac{2}{\Delta x} & i = j \\ -\frac{1}{\Delta x} & |i-j| = 1 \\ 0 & |i-j| \geq 2 \end{cases} \tag{1.14}$$

we obtain (1.7).

1.6 Completeness of the Sobolev Space

We introduce a function space to prove the existence of a minimizer. Let us confirm the definition of infimum. Let $A \subset \mathbb{R}$ be bounded from below. If $\alpha \in \mathbb{R}$ satisfies that

(i) $\alpha \leq x \quad (\forall x \in A)$;

(ii) $\forall \varepsilon > 0$, $\exists x_\varepsilon \in A$ s.t. $\alpha + \varepsilon > x_\varepsilon$,

we call α the infimum of A and write $= \inf A := \alpha$. By (ii), there exists $\{x_n\}_{n\in\mathbb{N}} \subset A$ s.t.

$$x_n \geq x_{n+1} \quad (\forall n \in \mathbb{N}), \quad x_n \to \alpha \quad (n \to \infty).$$

Similary, there exists $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ s.t.

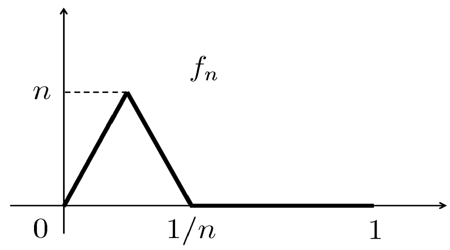
$$I(v_n) \geq I(v_{n+1}) \quad (\forall n \in \mathbb{N}), \quad \lim_{n\to\infty} I(v_n) = \inf_{v\in\mathcal{K}} I(v).$$

It is important to note that the sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ may not converge to a function v belonging to $\mathcal{K}$.

Example For each $n \in \mathbb{N}$, we define $f_n : [0,1] \to \mathbb{R}$ as follows

$$f_n(x) := \begin{cases} 2n^2 x & \text{if } x \in [0, 1/(2n)), \\ -2n^2 x + 2n & \text{if } x \in [1/(2n), 1/n), \\ 0 & \text{otherwise.} \end{cases}$$

The graph of f_n is shown below.



$\{f_n\}_{n\in\mathbb{N}}$ is a pointwise convergent function sequence. Pointwise means that for each $x \in \Omega$, there exists $f_x \in \mathbb{R}$ such that

$$f_n(x) \to f_x \quad (n \to \infty).$$

In this case, for each $x \in \Omega$, we define

$$f(x) := \lim_{n\to\infty} f_n(x) = 0.$$

Also, calculating the integra of f_n, we see that

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \frac{1}{2} \neq \int_\Omega f(x)dx.$$

On the other hand, if $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly convergent function sequence, it is possible to exchange the limit and the integral. Uniform convergence means that there exists a function $f : \Omega \to \mathbb{R}$ such that

$$\sup_{x\in\Omega} |f_n(x) - f(x)| \to 0 \quad (n \to \infty).$$

Then we have

$$\begin{aligned} \left| \int_\Omega f_n(x)dx - \int_\Omega f(x)dx \right| &\leq \int_\Omega |f_n(x) - f(x)|dx \\ &\leq \int_\Omega \sup_{\tilde{x}\in\Omega} |f_n(\tilde{x}) - f(\tilde{x})|dx \\ &= \sup_{x\in\Omega} |f_n(x) - f(x)||\Omega| \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

Hence

$$\lim_{n\to\infty}\int_\Omega f_n(x)dx = \int_\Omega f(x)dx.$$

We define the function space

$$L^2(\Omega) := \left\{ f \in \text{measurable}; \int_\Omega f(x)^2 dx < \infty \right\}.$$

$L^2(\Omega)$ is a linear space, and we define the L^2 inner product and the L^2 norm

$$(f,g) := \int_\Omega f(x)\cdot g(x)dx, \quad \|f\|_{L^2(\Omega)} := \sqrt{(f,f)} \quad (f,g \in L^2(\Omega)).$$

Then $L^2(\Omega)$ is a Banach space with the above norm.

Definition 1.5. *(weak derivative) For a function $u \in L^1_{loc}$ defined by*

$$L^1_{loc}(\Omega) := \left\{ f \in \text{measurable}; \forall K \Subset \Omega, \int_K |f(x)|dx < \infty \right\},$$

there exists a function $v \in L^1_{loc}$ such that

$$\int_\Omega \frac{\partial}{\partial x_i}\varphi(x)u(x)dx = -\int_\Omega \varphi(x)v(x)dx \quad (\forall \varphi \in C_0^\infty(\Omega)). \tag{1.15}$$

We call v the weak derivative of u, and we write $\partial u/\partial x_i := v$.

If u is a differentiable function, we obtain (1.15) from the integration by parts. Thus u has a weak derivative. We define the Sobolev space $W^{1,2}$.

Definition 1.6. *We define the Sobolev space*

$$W^{1,2}(\Omega) := \left\{ f \in \text{measurable}; f \in L^2(\Omega), f_{x_i} \in L^2(\Omega)\ (i = 1,2,\cdots,N) \right\},$$

where f_{x_i} is a weak derivative of f. Sometimes we write $H^1(\Omega)$ as $W^{1,2}(\Omega)$.

Let us show that $W^{1,2}(\Omega)$ equipped with the norm

$$\langle f,g\rangle := \int_\Omega (f\cdot g + \nabla f\cdot\nabla g)dx, \quad \|f\|_{W^{1,2}(\Omega)} := \sqrt{\langle f,f\rangle} \quad (f,g \in W^{1,2}(\Omega)),$$

is a Banach space. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ be a Cauchy sequence. From the definition of Cauchy sequence,

$$\|f_n - f_m\|_{W^{1,2}(\Omega)} \to 0 \quad (n,m\to\infty).$$

Then

$$\|f_n - f_m\|_{L^2(\Omega)} \le \|f_n - f_m\|_{W^{1,2}(\Omega)} \to 0$$

$$\left\|\frac{\partial f_n}{\partial x_i} - \frac{\partial f_m}{\partial x_i}\right\|_{L^2(\Omega)} \le \|f_n - f_m\|_{W^{1,2}(\Omega)} \to 0 \qquad i = 1, \cdots, N$$

as n and m to ∞. Therefore $\{f_n\}_{n\in\mathbb{N}}$ and $\{\partial f_n/\partial x_i\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $L^2(\Omega)$. From the completeness of $L^2(\Omega)$, there exists $f, v_i \in L^2(\Omega)$ such that

$$f_n \to f, \quad \frac{\partial f_n}{\partial x_i} \to v_i \quad \text{in } L^2(\Omega) \quad (n \to \infty).$$

Then we have from the Cauchy-Schwarz inequality,

$$\begin{aligned} \left|\int_\Omega \frac{\partial}{\partial x_i}\varphi(x) f_n(x)dx - \int_\Omega \frac{\partial}{\partial x_i}\varphi(x) f(x)dx\right| &\le \int_\Omega \left|\frac{\partial}{\partial x_i}\varphi(x)\right| |f_n(x) - f(x)|dx \\ &\le \left\|\frac{\partial \varphi}{\partial x_i}\right\|_{L^2(\Omega)} \|f_n - f\|_{L^2(\Omega)} \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \int_\Omega \frac{\partial}{\partial x_i}\varphi(x) f_n(x)dx = \int_\Omega \frac{\partial}{\partial x_i}\varphi(x) f(x)dx.$$

Similary, we obtain

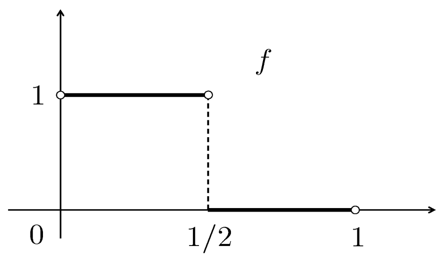
$$\lim_{n\to\infty} \int_\Omega \varphi(x) \frac{\partial}{\partial x_i} f_n(x)dx = \int_\Omega \varphi(x) v_i(x)dx.$$

Thu v_i is a weak derivative of f. From the above, $W^{1,2}(\Omega)$ is a Banach space.

Example 1.2. *Let $f\colon (0,1) \to \mathbb{R}$ be a function definded by*

$$f(x) := \begin{cases} 1 & x \in (0, 1/2) \\ 0 & x \in [1/2, 1) \end{cases}$$

f is not weakly differentiable.



Proof. We assume that f is weakly differentiable. Hence there exists $g \in L^1_{loc}(0, 1)$ such that

$$\int_0^1 \frac{\partial}{\partial x}\varphi(x) f(x)\, dx = -\int_0^1 \varphi(x) g(x)\, dx \quad (\forall \varphi \in C_0^\infty(0, 1)). \tag{1.16}$$

We calculate the left-hand side of (1.16)

$$\int_0^1 \frac{\partial}{\partial x}\varphi(x) f(x)\, dx = \int_0^{1/2} \frac{\partial}{\partial x}\varphi(x)\, dx = \varphi\left(\frac{1}{2}\right).$$

Therefore we have $g \equiv 0$ in $(0,1) \setminus \{1/2\}$. But this is a contradiction. □

1.7 Hilbert Space

Definition 1.7. *A Hilbert space H is a vector space endowed with an inner product $\langle \cdot, \cdot \rangle$ that is complete in the associated norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.*

Clearly, a Hilbert space H is also a Banach space.

Definition 1.8. *Let $\{u_n\} \subset H$ and $u \in H$. $\{u_n\}$ is said to converge weakly $u \in H$ ($u_n \rightharpoonup u$) if*

$$\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle$$

as $n \to \infty$, for each $\varphi \in H$.

Let X be a Banach space with a norm $\| \cdot \|_X$, and X^* be set of bounded linear functionals on X. From the definition, $f \in X^*$ satisfies

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \qquad u, v \in X,\ \alpha, \beta \in \mathbb{R},$$

and there exists $M > 0$ such that

$$|f(u)| \le M \|u\|_X \qquad \forall u \in X.$$

Then we can show that f is continuous. Indeed, for each $\{u_n\} \subset X$; $u_n \to u$ in X,

$$|f(u_n) - f(u)| = |f(u_n - u)| \le M \|u_n - u\| \to 0.$$

We can define norm for X^*.

Definition 1.9. *Let X be a Banach space and X^* be the set of bounded linear functionals on X. Then we define the norm on X^* as*

$$\|f\|_{X^*} := \sup_{u \in X, \|u\|_X = 1} |f(u)|.$$

X^* is called the dual space of X.

Obviously, X^* is also a Banach space. Similarly, we can define X^{**} as the set of bounded linear functionals on X^*.

Proposition 1.2. *Let X be a Banach space and X^{**} be the dual space of X^* where X^* is the dual space of X. Then $X^{**} \supset X$.*

Proof. Let $u \in X$ and we can define $u : X^* \to \mathbb{R}$,

$$u(f) := f(u) \quad (\forall f \in X^*)$$

Then we have that u is a linear functional of X^* and

$$|u(f)| \leq M\|u\|_X \leq \|u\|_X \|f\|_{X^*} \quad (\forall f \in X^*)$$

Hence $u \in X^{**}$. □

Definition 1.10. *Let X be a Banach space. If $X^{**} = X$, then we call X a reflexive Banach space.*

Definition 1.11. *Let X be a Banach space. X is called separable if there exists $\{\varphi_i\} \subset X$ such that $\overline{\{\varphi_i\}} = X$ where $\overline{A}$ is the closure of A in X.*

Theorem 1.1. *Let X be a separable Hilbert space and $\{f_n\} \subset X^*$. We assume that $\{f_n\}$ is bounded, i.e., there exists $M > 0$ such that $\|f_n\|_{X^*} \leq M$ $(\forall n \in \mathbb{N})$. Then there exists a subsequence $\{f_{n_j}\}_j$, $f \in X^*$ such that*

$$f_{n_j} \overset{*}{\to} f \quad (j \to \infty).$$

(i.e. $f_{n_j}(\varphi) \to f(\varphi)$ for all $\varphi \in X$).

Proof Since X is separable, there exists $\{\varphi_i\} \subset X$ such that $\overline{\{\varphi_i\}} = X$. Then we get

$$|f_n(\varphi_1)| \leq \|f_n\|_{X^*} \|\varphi_1\|_X \leq M\|\varphi_1\|_X \quad (\forall n \in \mathbb{N}),$$

hence there exists a subsequence $\{f_{1,n}(\varphi_1)\}_n$, $\alpha^1 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} f_{1,n}(\varphi_1) = \alpha^1.$$

Similarly, we can see that there exists a subsequence $\{f_{2,n}(\varphi_2)\}_n$, $\alpha^2 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} f_{2,n}(\varphi_2) = \alpha^2.$$

We write $\{f_{n_j}\}_j$ as $\{f_{n,n}\}$. Then $\{f_{n_j}\}_j$ satisfies that

$$f_{n_j}(\varphi_i) \to \alpha^i =: \tilde{f}(\varphi_i) \quad (j \to \infty)$$

for each $i \in \mathbb{N}$. Since $\overline{\{\varphi_i\}} = X$, for each $\varphi \in X$, there exists $\{\tilde{\varphi}_i\} \subset \{\varphi_i\}$ such that $\tilde{\varphi}_i \to \varphi$ in X $(i \to \infty)$. We define $f(\varphi) := \lim_{i\to\infty} \tilde{f}(\tilde{\varphi}_i)$. Then $f \in X^*$. □

Theorem 1.2. *(Riesz representation) Let H be a Hilbert space and $f \in H^*$. Then there exists a unique $v \in H$ such that*

$$f(u) = \langle v, u \rangle \quad (\forall u \in H).$$

Proof. First, we consider the finite dimensional case, $H = \mathbb{R}^n$. Then $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function and $\langle u, v \rangle = u \cdot v$ $(\forall u, v \in \mathbb{R}^n)$. We can rewrite the function f as a inner product. Next, we consider the infinite dimensional case. We define

$$\ker(f) := \{u \in H; f(u) = 0\}.$$

Since f is a linear, $\ker(f)$ is a linear subset and a closed set. In fact,

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) = 0$$

for each $u, v \in \ker(f)$, $\alpha, \beta \in \mathbb{R}$, and for $\{u_n\} \subset \ker(f)$; $u_n \to u_0$,

$$0 = \lim_{n\to\infty} f(u_n) = f\left(\lim_{n\to\infty} u_n\right) = f(u_0),$$

thus $u_0 \in \ker(f)$. Therefore, we get

$$H = \ker(f) \oplus \ker(f)^{\perp}, \tag{1.17}$$

In other words, for $u \in H$, there exsist a unique $u_1 \in \ker(f)$ and $u_2 \in \ker(f)^{\perp}$ such that $u = u_1 + u_2$. Let $u^{\perp} \in \ker(f)^{\perp}$, and we define

$$v := \frac{f(u^{\perp})u^{\perp}}{\langle u^{\perp}, u^{\perp} \rangle} \in \ker(f)^{\perp}.$$

From (1.17), for $u \in H$, there exists $w \in \ker(f)$ and $\alpha \in \mathbb{R}$ such that

$$u = w + \alpha u^{\perp}.$$

Then we calculate by using $\alpha = \langle u, u^{\perp} \rangle / \langle u^{\perp}, u^{\perp} \rangle$,

$$f(u) = f(w + \alpha u^{\perp}) = f(w) + \alpha f(u^{\perp}) = \frac{f(u^{\perp})}{\langle u^{\perp}, u^{\perp} \rangle}\langle u, u^{\perp} \rangle = \langle u, v \rangle,$$

for all $u \in H$. □