Porous medium equation

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1 Derivation for ideal gas in porous medium

Proposed by Boussinesq in 1903.

Darcy's law describes the flow of an ideal gas in a porous medium,

$$q = -\frac{\kappa}{\mu} \nabla P,$$

where q is the volume flux, κ is the permeability of the medium, μ is the dynamic viscosity and P is the pressure.

State equation of ideal gas describes the relation between pressure P, density ρ and temperature T of an ideal gas,

$$P = \rho \frac{R}{M} T,$$

where R is the ideal gas constant and M the molar mass.

Let Ω be a test volume with a nice boundary. The law of conservation of energy states the change of mass in Ω is equal to the amount of mass that enters Ω. Since there are no sources, the entering mass is given only by the mass flux $q\rho$ of mass through the boundary. Thus the conservation of mass reads

$$\frac{d}{dt} \int_\Omega \rho \, dx = -\int_{\partial\Omega} \rho q \cdot \eta \, dS,$$

where η is the outer unit normal to $\partial\Omega$. Using the Darcy's law and the state equation, we get

$$q = -\frac{\kappa}{\mu} \nabla P = -\frac{\kappa}{\mu} \frac{R}{M} \nabla(\rho T).$$

Now we assume an isothermal process, so T is constant, we denote $c = \frac{\kappa}{\mu} \frac{R}{M} T$ and

$$q = -c \nabla \rho.$$

Hence the conservation of mass is

$$\frac{d}{dt} \int_\Omega \rho \, dx = c \int_{\partial\Omega} \rho \nabla \rho \cdot \eta \, dS.$$

Note that $\rho\nabla\rho = \frac{1}{2}\nabla\left(\rho^2\right)$, and so

$$\frac{d}{dt}\int_\Omega \rho\, dx = \frac{c}{2}\int_{\partial\Omega} \nabla\left(\rho^2\right)\cdot\eta\, dS.$$

The divergence theorem than yields

$$\frac{d}{dt}\int_\Omega \rho\, dx = \frac{c}{2}\int_{\Omega} \Delta\left(\rho^2\right)\, dx.$$

Since this is valid for all test volumes Ω, we get finally

$$\frac{d\rho}{dt} = \frac{c}{2}\Delta\left(\rho^2\right).$$

2 Porous medium equation

Porous medium equation in density variable $\rho \geq 0$, $m > 1$

$$\begin{aligned} \rho_t &= \Delta\left(\rho^m\right), && \text{in } \mathbb{R}^n \times (0,\infty),\\ \rho|_{t=0} &= \rho_0, && \rho_0 \in C(\mathbb{R}^n). \end{aligned}$$

$m > 1$.

In the pressure variable $u \geq 0$, $a = m - 1 > 0$, change of variables

$$u = \frac{a+1}{a}\rho^a.$$

$$\begin{aligned} u_t &= au\Delta u + |\nabla u|^2, && \text{in } \mathbb{R}^n \times (0,\infty),\\ u|_{t=0} &= u_0, && u_0 \in C(\mathbb{R}^n). \end{aligned}$$

PME is used for

- $m = 2$ – ideal gas in porous media
- $m \geq 2$ – compressible fluids through porous media
- $m = 4$ – spread of a thin viscous film under gravity
- $m \approx 6$ – thermal propagation in plasma

It is known, see for example [3], that if $\rho_0 \in L^1(\mathbb{R}^n)$, there exists a unique nonnegative weak solution, and for each t it has compact support that increases with t. Hence there exists an interface or free boundary separating regions where $\rho > 0$ and $\rho = 0$. The solution is C^∞-smooth in its positivity set, but the interface might not be a smooth surface.

Even real analytic initial data is not sufficient to ensure the existence of a classical solution of the PME for all $t > 0$.

3 Viscosity solutions

Let

$$Q = \mathbb{R}^n \times (0, T).$$

A continuous and nonnegative function u defined in Q is a **viscosity subsolution** of PME iff for every $\phi \in C^{2,1}(Q)$ that touches u from above at a point (x_0, t_0) the inequality

$$\phi_t \leq a\phi\Delta\phi + |\nabla\phi|^2$$

holds at (x_0, t_0).

We say that ϕ **touches** u from above at a point (x_0, t_0) if $\phi - u$ reaches a local minimum zero in a parabolic neighborhood of (x_0, t_0), i.e., there exists a cylinder $R = B_r(x_0) \times (t_0 - \tau, t_0]$, such that $u \leq \phi$ in R and $u = \phi$ at $x_0, t_0)$.

A function $u \in C(Q)$, $u \geq 0$ us a **classical free boundary solution** if

(a) it is positive in an open set $P(u) \subset Q$ where it is smooth and solves PME in the classical sense,

(b) the boundary of $P(u)$, $\Gamma = \partial P \cap Q$, free boundary, is a smooth hypersurface in space-time and $u \in C^{2,1}(P \cup \Gamma)$.

(c) on Γ the following dynamic condition holds

$$v_n = |\nabla u|,$$

where v_n is the normal speed of advance of the free boundary.

In the case $\nabla u \neq 0$ on Γ, we call the solution a **classical moving free-boundary solution**.

(c) follows from the fact that on Γ the PME has the form

$$u_t = |\nabla u|^2$$

and thus the free boundary moves with the speed

$$v_n = \frac{u_t}{|\nabla u|} = |\nabla u|.$$

A function $u \in C(Q)$, $u \geq 0$ is a **viscosity supersolution** of PME iff the two following conditions are satisfied:

(a) For every funciton $\phi \in C^{2,1}(Q)$ that touches u from below at a point (x_0, t_0) where $u(x_0, t_0) > 0$ the inequality

$$\phi_t \geq a\phi\Delta\phi + |\nabla\phi|^2$$

holds at (x_0, t_0).

(b) every classical moving free-boundary solution that lies below u at a time $t = t_1 \geq 0$ cannot cross u at a later time $t_2 > t_1$.

A **viscosity solution** of PME is a continuous and nonnegative function defined in Q which is at the same time a sub- and a supersolution. A viscosity solution of the Cauchy problem is a viscosity solution of the PME which is also continuous at $t = 0$ and satisfies the initial condition.

4 Maximum principle

For the general proof, see [1], Theorem 14.1, page 363.

Define

$$Pu = au\Delta u + |\nabla u|^2 - u_t.$$

Consider the PME on the set $\Omega_T = \Omega \times (0, T)$, where Ω is a bounded open set. Define $\Gamma_T = \Omega \times \{0\} \cup \partial\Omega \times [0, T]$.

Theorem 1. *Assume $u, v \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ such that $Pu \geq Pv$ in Ω_T and $u \leq v$ in Γ_T. Then $u \leq v$ in $\overline{\Omega_T}$.*

Proof. Δv is continuous on a compact set, and hence it is bounded. Thus we can find $K > a \sup_{\Omega_T} |\Delta v|$. Define

$$\tilde{u} = e^{-Kt} u, \qquad \tilde{v} = e^{-Kt} v.$$

Suppose that $u > v$ somewhere in Ω_T. That means that $\tilde{u} - \tilde{v}$ has a positive maximum at some point $(x_0, t_0) \in \overline{\Omega_T} \setminus \Gamma_T$. Define $E = e^{Kt_0}$. Note that at (x_0, t_0), we have $\Delta u = E\Delta\tilde{u}$, $\Delta v = E\Delta\tilde{v}$, $\nabla u = E\nabla\tilde{u}$ and $\nabla v = E\nabla\tilde{v}$. As (x_0, t_0) is a maximum of $\tilde{u} - \tilde{v}$, we also have

$$\nabla\tilde{u} = \nabla\tilde{v}, \quad \Delta\tilde{u} \leq \Delta\tilde{v}, \quad \tilde{u}_t \geq \tilde{v}_t.$$

Differentiate

$$\tilde{u}_t = -Ke^{-Kt} u + e^{-Kt} u_t,$$

so at $t = t_0$,

$$u_t = E(\tilde{u}_t + K\tilde{u}), \qquad v_t = E(\tilde{v}_t + K\tilde{v}).$$

So we can evaluate at (x_0, t_0),

$$\begin{aligned} Pu &= aE\tilde{u}\Delta E\tilde{u} + |\nabla E\tilde{u}|^2 - E(\tilde{u}_t + K\tilde{u}) \\ &\leq aE\tilde{u}\Delta E\tilde{v} + |E\nabla v|^2 - E(\tilde{v}_t + K\tilde{u}) \\ &< aE\tilde{v}\Delta E\tilde{v} + |E\nabla v|^2 - E(\tilde{v}_t + K\tilde{v}) \\ &= Pv. \end{aligned}$$

The sharp inequality is possible as $|a\Delta E\tilde{v}| = a|\Delta v| \leq a \sup |\Delta v| < K$ and $E\tilde{u} > E\tilde{v}$. Thus we get a contradiction with the assumption $Pu \geq Pv$. ☐

5 Exact solutions

Traveling waves for the PME (always in the pressure variable) takes the form

$$T(x, t; c) = c(x \cdot e + ct)_+,$$

where $c \in \mathbb{R} \setminus \{0\}$, $e \in \mathbb{R}^n$ unitary vector.

Barenblatt solutions, family of source-type solutions, in the form

$$B(x,t;\tau,C) = (t+\tau)^{-\lambda na}\left(C - \kappa\frac{x^2}{(t+\tau)^{2\lambda}}\right)_+,$$

where $\lambda = (an+2)^{-1}$, $\kappa = \lambda/2$ and C and $\tau > 0$ are arbitrary. Source-type because $B \to \delta$ as $t \to -\tau$.

Curved traveling wave supersolutions of the form

$$u = A(|x| + ct - B)_+,$$

with constants $A, B, c > 0$ in a domain $\mathcal{R} = \{|x| < R, -T < t < 0\}$. the function is nontrivial in $\mathcal{R}$ if $R > B$, it has constant gradient $|\nabla u| = A$ in the positivity set and constant advance speed of the free boundary $v_n = c$. Support u does not go into the ball $B_{R/2}(0)$ if $B > R/2$. If moreover

$$\frac{c}{A} > 1 + 2a(n-1)\frac{R-B}{R},$$

u is a classical moving free-boundary supersolution.

6 Existence of viscosity solutions

For positive initial data $u_0 \in C(\mathbb{R}^n)$, $u_0 \geq \varepsilon > 0$, PME admits a unique classical solution $u \in C^\infty(Q)$, $u \geq \varepsilon$.

Lemma 2. *A positive $u \in C^2(Q)$ is a viscosity solution of PME iff it is a classical solution.*

Construction of viscosity solution for general data, $u_0 \in C(\mathbb{R}^n)$. For that we may solve the Cauchy problem with data $u_{0,\varepsilon} = u_0 + \varepsilon$ and obtain a classical solution $u_\varepsilon(x,t) \geq \varepsilon$. By the maximal principle, $u_\varepsilon \geq u_{\varepsilon'}$ if $\varepsilon \geq \varepsilon'$. So there is a limit

$$U(x,t) = \lim_{\varepsilon \to 0+} u_\varepsilon(x,t).$$

Lemma 3. *U is a viscous solution of the initial problem. It is moreover the maximal viscosity solution of the problem.*

Proof. Regularity theory for weak solutions of the PME the family u_ε is bounded in $C^{0,\alpha}$ for some $\alpha > 0$, hence, there is a subsequence that it converges locally uniformly to U, which implies that U is a viscosity solution. By comparing any other viscosity solution with u_ε, we see that U is maximal. □

7 Sup and inf convolutions

This concept is used in [2]. Given a viscosity subsolution u and a radius $r > 0$ we define the function $\overline{u}_r$ as

$$\overline{u}_r(x,t) = \sup_{B_r(x,t)} u(y,\tau),$$

where $B_r(x,t) = \{(y,\tau) : |y-x|^2 + (t-\tau)^2 \le r\}$ is a ball in space-time. Given a viscosity supersolution u we define

$$\underline{u}_r(x,t) = \sup_{B_r(x,t)} u(y,\tau).$$

Lemma 4. *The function $\overline{u}_r$ is a viscosity subsolution, the function $\underline{u}_r$ is a viscosity supersolution of PME for $t \ge r$. Moreover, the positivity set of $\overline{u}_r$ has the interior ball property on the boundary with radius r and the support of $\underline{u}_r$ has the exterior ball property with radius r. At the points of the boundary of the support of u where these balls are centered we have the complementary statements: an exterior ball in the first case, an interior ball in the second.*

Proof. Let $\phi \in C^{2,1}$ such that ϕ touches $\underline{u}_r$ from below at $P_0 = (x_0, t_0)$, i.e. there is some parabolic neighborhood $R = B_\delta(x_0) \times (t_0 - \tau, t_0]$ where $\phi \le \underline{u}_r$ and $\phi(x_0,t_0) = \underline{u}_r(x_0,t_0)$. As $B_r(x_0,t_0)$ is compact and u is continuous, there is $(x_1,t_1) \in B_r(x_0,t_0)$ such that $\underline{u}_r = u(x_1,t_1)$. Since

$$\phi(x,t) \le \underline{u}_r(x,t) \le u(x + x_1 - x_0, t + t_1 - t_0) \qquad \text{for all } (x,t) \in R,$$

we see that ϕ touches u from below at (x_0,t_0). As u is a supersolution, we thus have $\phi_t \ge a\phi\Delta\phi + |\nabla\phi|^2$. □

8 Main result

Theorem 5. *The initial value problem is well-posed in the class of viscosity solutions for continuous and nonnegative initial data. The viscosity solution coincides with the weak solution.*

Existence is given by the weak theory (Lemma 3) by taking the limit

$$U(x,t) = \lim_{\varepsilon \to 0+} u_\varepsilon(x,t).$$

The only question is the uniqueness of the maximal viscosity solution.

For that we need a comparison of viscosity solutions.

We say that a pair of function u_0, v_0, is **strictly ordered** if

(a) the support of u_0, $\operatorname{supp} u_0$, is a compact subset of $\mathbb{R}^n$ and

$$\operatorname{supp} u_0 \subset (\operatorname{supp} v_0)^O,$$

(b) inside the support of u_0 the functions are strictly ordered

$$u_0(x) < v_0(x).$$

Theorem 6. *Suppose u is a viscosity subsolution and v is a viscosity supersolution of PME and they have strictly ordered initial data. Then the solutions remain ordered for all time: $u \le v$ in all $\mathbb{R}^n \times (0,\infty)$.*

Proof. The proof consists of a couple ideas:

(a) sup and inf functions Define functions

$$W(x,t) = \inf_{B_{r-\delta t}} v(y,s),$$

well defined in $\mathbb{R}^n \times (r,T)$ with $T = r/\delta$, $\tau = r/(1+\delta)$, and

$$Z(x,t) = \sup_{B_r} u(y,s),$$

well defined in $\mathbb{R}^n \times (r,T)$. When $r, \delta > 0$ are small enough and $T \leq r/\delta$:

(i) W is a supersolution and Z is a subsolution of PME in $Q = \mathbb{R}^n \times (r,T)$,

(ii) $Z(\cdot, r)$ and $W(\cdot, r)$ are strictly ordered.

(a) is a consequence of Lemma 4. (b) is the consequence of the fact that Z and W are continuous, u_0 has a compact support and also the fact that the free boundary of u cannot expand in a discontinuous way.

(b) free boundary contact

(c) linear boundary behavior

□

[1] Gary M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, 1996.

[2] Luis Caffarelli and Juan Luis Vázquez, *Viscosity Solutions for the Porous Medium Equation*, Differential Equations: La Pietra 1996, Proceeding of Symposia in Pure Mathematics, vol. 65, American Mathematical Society, 1999.

[3] Ki-ahm Lee and Juan Luis Vázquez, *Geometrical Properties of Solutions of the Porous Medium Equation for Large Times*, Indiana U. Math. J. **52** (2003), no. 4, 991–1015.