Explicit viscosity solution of the crystalline curvature flow with fattening

Norbert Požár
Faculty of Mathematics and Physics
Institute of Science and Engineering, Kanazawa University

Abstract

We give a simple explicit example of a solution of the level set formulation of the crystalline curvature flow in two dimensions whose one level set fattens, leading to nonuniqueness of the flow, and prove that it is the unique viscosity solution. The motivation of this note is to illustrate the notion of viscosity solutions introduced recently by Y. Giga and the author.

1 Introduction

The purpose of this short note is to illustrate the notion of viscosity solutions of the crystalline mean curvature flow introduced by Y. Giga and the author in [13, 14]. We will construct an explicit viscosity solution with a polygonal figure 8 initial data as an example of fattening in this problem. Fattening refers to the essential nonuniqueness present in the mean curvature flow where multiple evolutions are possible. A well-known example of this phenomenon in the context of the standard mean curvature flow in two dimensions is the initially connected, self-intersecting curve resembling the figure 8. During the evolution, the curve either splits at the self-intersection into two disconnected components that contract to two distinct points, or it separates at the self-intersection and evolves as a simple closed curve that contracts to a single point; see [7, 8].

The level set formulation of the mean curvature flow has a unique solution even when the evolution of the curve is not unique. The level set describing the evolution of the curve however develops a nonempty interior. That is, the level set fattens. Examples of fattening are discussed for instance in [7, 8, 15, 17, 18], but none of them are given by an explicit formula.

On the other hand, the solutions of the crystalline curvature flow in two dimensions, reviewed in Section 2, are in general polygonal curves whenever the initial curve is polygonal, and so we can expect that the crystalline algorithm [22, 23] for tracking such evolutions will give an explicit solution. One such solution is constructed in Section 3, and the proof that it is the unique viscosity solution of the level set formulation is presented in Section 3.2.

2 The crystalline mean curvature flow

The same way that one can understand the standard curvature flow as the formal gradient flow of the curve length, one can consider the formal gradient flow $t \mapsto E_t$ of the anisotropic perimeter

$$P_\sigma(E) := \int_{\partial^* E} \sigma(\nu)\, d\mathcal{H}^1 \tag{2.1}$$

defined for sets of finite perimeter $E \subset \mathbb{R}^2$, where ν is the outer unit normal, $\sigma : S^1 \to (0, \infty)$ expresses the anisotropy of the perimeter, and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

This formally yields the geometric evolution

$$V = -\kappa_\sigma, \tag{2.2}$$

where V is the normal velocity of the evolving boundary ∂E_t. The term κ_σ is the anisotropic mean curvature of ∂E_t corresponding to the anisotropy σ. It is the first variation of the anisotropic perimeter P_σ [2, 4]. When the anisotropy σ is nonsmooth so that the Wulff shape, that is, the set with the smallest anisotropic perimeter for a given volume, is a convex polytope, it is referred to as the crystalline mean curvature flow. This problem was introduced by Angenent and Gurtin [3] and Taylor [21] as a model of growth of crystals. Solutions of the flow in two dimensions are often evolving polygonal curves. Furthermore, the velocity of each segment can be found quite easily as a constant inversely proportional to the length of the segment; see (3.5). This motivates the crystalline algorithm [21–23]. It is a first order ODE system for the positions of the individual segments. This algorithm provides an efficient way to find the evolution before the onset of singularities. When singularities occur, the evolution can sometimes be continued in a natural way [1], but there are many examples where a much more care must be taken as in the case of collisions and topological changes [16, 19].

The viscosity solutions [10–14] provide one way to select a unique continuation when no fattening occurs. It is also known that the crystalline algorithm generates a viscosity solution when no singularities occur [9, 11]. Another notion of generalized solutions was developed in [5, 6]. It is defined in terms of the signed distance function to the evolving curve (surface) and relies for existence on the minimizing movements algorithm [2]. For a more detailed discussion see [6, 13] and the reference therein.

The level set equation for the crystalline curvature flow can be formally written as

$$u_t = |\nabla u| \operatorname{div} \nabla\sigma(\nabla u) \qquad \text{in } \mathbb{R}^2 \times (0, \infty), \tag{2.3}$$

so that every level set of the auxiliary function u defines a curve that evolves under the geometric flow (2.2) [12]. The term $\operatorname{div} \nabla\sigma(\nabla u)$ must be properly understood in the sense of a subdifferential of the anisotropic total variation energy

$$\mathcal{E}(\psi) := \int_{\mathbb{R}^2} \sigma(\nabla\psi)\, dx.$$

Here $\sigma : \mathbb{R}^2 \to [0, \infty)$ is the positively one-homogeneous extension of σ in (2.1) given as $\sigma(p) = |p|\sigma(p/|p|)$, $p \neq 0$, and it is assumed to be convex. Any such function we shall call an *anisotropy*. If σ is furthermore given as the maximum of a finite number of linear functions,

$$\sigma(p) := \max_i x_i \cdot p,$$

we shall call it a *crystalline* anisotropy. In this case the vector field $\nabla\sigma(\nabla u)$ is in general not continuous even if the level sets of u are smooth curves and therefore one cannot expect to be able to express $\operatorname{div} \nabla\sigma(\nabla u)$ as a function. In fact, this operator becomes nonlocal on flat parts parallel to the sides of the Wulff shape.

In this note we will focus on the crystalline anisotropy given by the ℓ^1 norm: $\sigma(p) := \|p\|_1 = |p_1| + |p_2|$. Clearly we can take x_i to be the four points $(1, 1)$, $(-1, 1)$, $(1, -1)$ and $(-1, -1)$. These happen to also be the vertices of the Wulff shape $\mathcal{W}$ corresponding to σ,

$$\mathcal{W} = \{x : \sigma^\circ(x) \leq 1\} = \{x : \|x\|_\infty \leq 1\},$$

where $\sigma^\circ : \mathbb{R}^2 \to [0, \infty)$ is the convex polar

$$\sigma^\circ(x) := \max\{x \cdot p : \sigma(p) \leq 1\} = \|x\|_\infty,$$

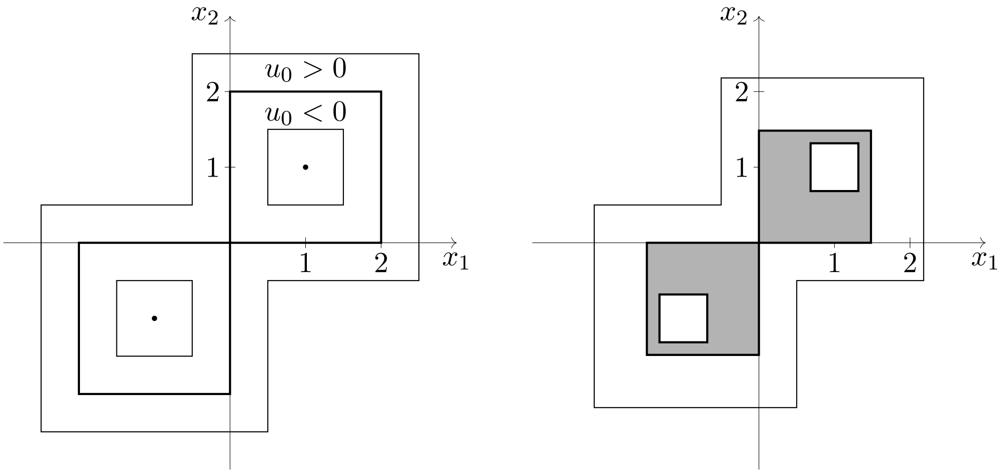

Figure 1: Left: Level sets $\{u_0 = \theta\}$ of the initial data for $\theta = -1$ (two points), $-\frac{1}{2}$ (two squares), 0 (thick figure '8'), and $\frac{1}{2}$. Right: Level sets $\{u(\cdot, t) = 0\}$ (gray region) and $\{u(\cdot, t) = \frac{1}{2}\}$ at $t = 0.45$.

see [20]. The Wulff shape is the set with the smallest anisotropic perimeter given the volume and in this case it happens to be a square.

It is important to understand for what kind of curves we can define the crystalline curvature. In the case of the ℓ^1 anisotropy, we can for instance consider the following class: A curve is *regular* for the anisotropy $\sigma = \|\cdot\|_1$ if it is a Lipschitz regular boundary of an open set and consists only of axes-aligned segments with positive length; see also [23].

3 The example of fattening

We consider the initial data

$$u_0(x) = \min\Big(\|x - (1,1)\|_\infty - 1, \|x + (1,1)\|_\infty - 1, 1\Big), \qquad x \in \mathbb{R}^2,$$

for the level set equation (2.3). Note that $-1 \le u_0 \le 1$. The zero level set $\{u_0 = 0\}$ of u_0 is the boundary of the union of two axes-aligned squares of side-length 2 touching at the origin, centered at $(1,1)$ and $(-1,-1)$, respectively; see Figure 1. This is the polygonal figure '8' shape.

Other level sets $\{u_0 = \theta\}$ for $-1 < \theta < 1$ are boundaries of unions of two squares of side-length $2 + 2\theta$ with the same centers as above, but the squares are disjoint for $-1 < \theta < 0$, while the intersection has a nonempty interior for $\theta > 0$.

The evolution of the initial polygonal curves $\{u_0 = \theta\}$ is therefore quite clear for $\theta \in \Theta := (-1, 0) \cup (0, 1)$ by the crystalline algorithm since these initial curves are *regular*, see above, with respect to the anisotropy $\sigma = \|\cdot\|_1$. From these evolving curves, we will construct a level set function $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ and show that it is the unique viscosity solution of (2.3) with initial data u_0.

3.1 Crystalline algorithm

We will express the evolving curves as the boundaries of evolving open sets $E_t^\theta \subset \mathbb{R}^2$ for $\theta \in \Theta$, with initial data $E_0^\theta = \{u_0 < \theta\}$.

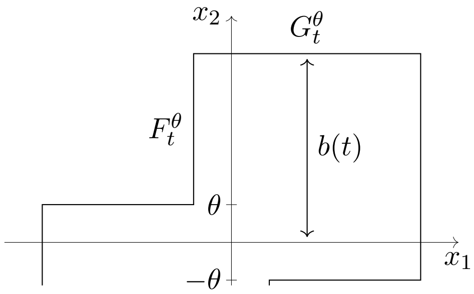

Figure 2: The boundary of E_t^θ for $\theta \in (0,1)$ and $0 \le t < t^\sharp$.

Case $-1 < \theta < 0$

$E_0^\theta := \{u_0 < \theta\}$ is the union of two disjoint squares

$$E_0^\theta = Q_{1+\theta}(1,1) \cup Q_{1+\theta}(-1,-1), \tag{3.1}$$

where $Q_r(c) := \{x : \|x-c\|_\infty < r\}$. Each square can be evolved independently. If we denote the half-length of their sides as $a = a(t)$, $a(0) = 1+\theta$, the crystalline curvature on the facet is given as

$$\kappa_\sigma = \frac{1}{a}.$$

This follows from a general formula $\kappa_\sigma =$ (length of Wulff shape facet)/(length of facet), which is rigorously written in (3.5) below. The velocity in the outer normal direction of each side of the square happens to be the derivative a'. Therefore under the crystalline algorithm, a satisfies

$$\begin{cases} a' = -\dfrac{1}{a}, & t > 0, \\ a(0) = 1+\theta, \end{cases}$$

with exact solution $a(t) = \sqrt{(1+\theta)^2 - 2t}$ for $t < t^* := \frac{(1+\theta)^2}{2}$, where $t^* = t^{*,\theta}$ is the extinction time, that is, the time such that $a(t) \searrow 0$ as $t \nearrow t^*$. We continue the solution as $a(t) = 0$ for $t \ge t^*$.

Remark 3.1. In two dimensions, the flow with $\sigma = \|\cdot\|_1$ reduces the area of each connected component of the interior of the curve at a constant rate -8. Since the initial area of each component in this case is $4(1+\theta)^2$, the area becomes 0 exactly at t^*.

We define

$$E_t^\theta := Q_{a(t)}(1,1) \cup Q_{a(t)}(-1,-1), \qquad -1 < \theta < 0,\ t \ge 0.$$

Note that $E_t^\theta = \emptyset$ for $t \ge t^*$.

Case $0 < \theta < 1$

$E_0^\theta = \{u_0 < \theta\}$ is again a union of two squares as in (3.1), however, this time they are not disjoint. In fact E_0^θ has 8 facets; see Figure 1. By the symmetry with respect to both diagonals, we need to only consider the two of them in $\{(x_1, x_2) : x_2 > |x_1|\}$, namely

$$F_0^\theta := \{(-\theta, x_2) : \theta < x_2 < 1+\theta\}, \quad G_0^\theta := \{(x_1, 1+\theta) : -\theta < x_1 < 1+\theta\},$$

see Figure 2. The crystalline curvature is 0 on F_0^θ by (3.5) and a positive constant depending on the length of the facet on G_0^θ. Therefore F_0^θ does not move, only shortens, while G_0^θ moves in the direction of E_0^θ. We introduce $b = b(t)$ to be the distance to the x_1-axis of the facet G_t^θ,

$$F_t^\theta := \{(-\theta, x_2) : \theta < x_2 < b(t)\}, \quad G_t^\theta := \{(x_1, b(t)) : -\theta < x_1 < b(t)\}.$$

We define the time $t^\sharp = t^{\sharp,\theta}$ as the time when $b(t^\sharp) = \theta$, and $F_{t^\sharp}^\theta$ vanishes. At this time $E_{t^\sharp}^\theta$ is the square $Q_\theta(0,0)$. After that the evolution is self-similar (homothetic) as in the previous case.

Let us find an expression for $b(t) = b^\theta(t)$. The crystalline curvature on G_t^θ is

$$\kappa_\sigma = \frac{2}{b(t) + \theta}.$$

As before, b' is the normal velocity of the facet G_t^θ and therefore b is the solution of

$$\begin{cases} \quad b' = -\dfrac{2}{b+\theta}, & t > 0, \\ b(0) = 2 + \theta, \end{cases}$$

with formula

$$b(t) = \sqrt{(2+2\theta)^2 - 4t} - \theta.$$

We have $b(t^\sharp) = \theta$ at $t^\sharp = t^{\sharp,\theta} := 1 + 2\theta$. After this time, E_t^θ is the square

$$E_t^\theta = Q_{c(t)}(0),$$

where $c(t) = c^\theta(t) = \sqrt{\theta^2 - 2(t - t^\sharp)}$, $t^\sharp \le t < t^\sharp + \frac{\theta^2}{2}$. At $t^* := t^\sharp + \frac{\theta^2}{2} = 1 + 2\theta + \frac{1}{2}\theta^2$ the set vanishes.

We have

$$E_t^\theta = \begin{cases} A_t^\theta \cup (-A_t^\theta), & 0 \le t < t^\sharp = 1 + 2\theta, \\ Q_{c(t)}(0), & t^\sharp \le t < t^* := t^\sharp + \frac{1}{2}\theta^2 = 1 + 2\theta + \frac{1}{2}\theta^2, \\ \emptyset, & t \ge t^*, \end{cases}$$

with $A_t^\theta = \{(x_1, x_2) : -\theta < x_1, x_2 < b(t)\}$. As in Remark 3.1, we can find $t^\sharp$ and t^* by considering the area of E_0^θ and the fact that the area of E_t^θ decreases with a constant rate -8.

We clearly have the following monotonicity:

$$E_t^\eta \subset E_t^\theta \subset E_s^\theta \qquad \text{for } \eta \le \theta, 0 \le s \le t, \quad \eta, \theta \in \Theta.$$

In words, the sets are nonincreasing in time for each θ, and they form a monotonically increasing family in θ. This monotonicity of E_t^θ is clear from the explicit formulas, but it is also a consequence of the comparison principle for the crystalline algorithm. In fact, it is rather easy to see that

$$\operatorname{dist}(E_t^\eta, (E_t^\theta)^{\mathsf{c}}) \ge \theta - \eta = \operatorname{dist}(E_0^\eta, (E_0^\theta)^{\mathsf{c}}) \qquad \text{whenever } \eta \le \theta, \ \eta, \theta \in \Theta. \tag{3.2}$$

Moreover, due to the semigroup property we can always compare the evolving set with the homothetic solution starting from $Q_\delta(c)$ centered at any point c.

Lemma 3.2. *Whenever for some $c \in \mathbb{R}^2$, $t \ge 0$ and $\delta > 0$ we have $x \in E_t^\theta$ for all $x \in \mathbb{R}^2$ with $\|x - c\|_\infty < \delta$, we can conclude that $c \in E_s^\theta$ for $t \le s < t + \frac{\delta^2}{2}$.*

We will now define the level set function

$$u(x,t) := \inf\Big(\big\{\theta \in (-1,0) \cup (0,1) : x \in E_t^\theta\big\} \cup \{1\}\Big). \tag{3.3}$$

It is easy to see that $-1 \le u \le 1$ on $\mathbb{R}^2 \times [0,\infty)$, $u(x,t) = 1$ for $\|x\|_\infty \ge 3$ or $t \ge \lim_{\theta\to 1-} t^* = 3.5$, and u is nondecreasing in time. Finally, u is Lipschitz in space with Lipschitz constant 1 by (3.2).

In fact, we can write a rather explicit formula for u. To simplify the situation, we again take advantage of the symmetries of u guaranteed by the symmetries of the initial data and give the values of $u(\cdot,t)$ only on the set $\{x_2 \ge |x_1|\}$. Let us define

$$\xi(x_2,t) := \begin{cases} \sqrt{x_2^2 + 2t + 2} - 2, & x_2 < (t-1)/2,\\ \frac{1}{3}\left(x_2 + \sqrt{4(x_2-1)^2 + 12t} - 4\right), & \text{otherwise.}\end{cases}$$

It is not difficult to check that ξ is 1-Lipschitz continuous since at $x_2 = (t-1)/2$ both branches are equal to x_2. Note also that $\xi(x_2,t) < 0$ for $0 \le x_2 < 2\sqrt{1-t}$ and $0 \le t < 1 = t^{\sharp,0}$.

For $x_2 < (t-1)/2$, $\theta = \xi(x_2,t)$ is the solution of $b^\theta(t) = x_2$, that is, the point x might lie on the edge G_t^θ. Otherwise $\theta = \xi(x_2,t)$ is a solution of $c^\theta(t) = x_2$, in which case x lies on the top edge of $Q_{c(t)}$. Let us combine this with the possibility that x lies on the edge F_t^θ, which does not move:

$$v(x,t) := \min\Big(\max(-x_1, \xi(x_2,t)), 1\Big).$$

On the other hand, x can lie on the boundary of $Q_{a(t)}(1,1)$ or in the fattened zero set, which we express by

$$w(x,t) := \min\left(\sqrt{\|x-(1,1)\|_\infty^2 + 2t} - 1, 0\right).$$

Then we can write u for $x_2 \ge |x_1|$ as

$$u(x,t) = \begin{cases} v(x,t), & v(x,t) > 0,\\ w(x,t), & \text{otherwise.}\end{cases} \tag{3.4}$$

u can be extended to the rest of $\mathbb{R}^2$ by symmetry.

3.2 Viscosity solution

We claim that u is the unique viscosity solution of (2.3) with initial data u_0.

The definition of viscosity solutions introduced in [13, 14] can be written in $n = 2$ with $\sigma(p) = \|p\|_1 = |p_1| + |p_2|$ in the following simplified way. First we introduce *admissible support* functions as the Lipschitz functions on $\mathbb{R}$ whose zero level set $\{\psi = 0\}$ is the union of finite number of bounded closed intervals of positive length. Here we should note that the definition of admissible functions in [14] is potentially more general but the important point is that the class considered here still gives the density result [14, Theorem 1.3]: For every Lipschitz continuous function ψ on $\mathbb{R}$ with $\{\psi = 0\}$ compact and any $\rho > 0$ there exists an admissible support function ζ with $\operatorname{sign}\psi(x) \le \operatorname{sign}\zeta(x) \le \sup_{|y-x|<\rho} \operatorname{sign}\psi(y)$. Note that $\operatorname{sign} s := -1, 0, 1$ whenever $s < 0$, $s = 0$, $s > 0$, respectively. This result is a simple consequence of the structure of open subsets of the real line.

For such an admissible support function ψ, we will define

$$\Lambda[\psi](x) := \frac{\chi(x)}{L(x)} \qquad x \in \{\psi = 0\}, \tag{3.5}$$

where $L(x)$ is the length of the connected component of $\{\psi = 0\}$ that contains x, and $\chi(x)$ is the sum of the signs of the two boundary points of the connected component. We define the sign of the boundary point as 1 if $\psi \geq 0$ in its neighborhood, and -1 otherwise. Therefore the possible values of $\chi(x)$ are -2, 0 and 2.

Definition 3.3. *We say that an upper semi-continuous function u is a viscosity subsolution of* (2.3) *whenever the following conditions are satisfied:*

(i_1) If $\varphi(x,t) = \psi(x_1) + f(x_2) + g(t)$, where ψ is an admissible support function and f, $g \in C^1(\mathbb{R})$, $f'(\hat{x}_2) \neq 0$, and $u - \varphi(\cdot - he_1)$ has a global maximum at $(\hat{x}, \hat{t})$ for all $h \in \mathbb{R}$, $|h|$ small, and $\hat{x}_1 \in \operatorname{int}\{\psi = 0\}$, then

$$g'(\hat{t}) - |f'(\hat{x}_2)|\Lambda[\psi](\hat{x}_1) \leq 0.$$

(i_2) Same as (i_1) *but with subscripts 1 and 2 exchanged.*

(ii) If $\varphi(x,t) = \psi(x) + g(t)$, where $\psi \in Lip(\mathbb{R}^2)$ and $g \in C^1(\mathbb{R})$, and $u - \varphi(\cdot - h)$ has a global maximum at $(\hat{x}, \hat{t})$ for all $h \in \mathbb{R}^2$, $|h|$ small, and $\hat{x} \in \operatorname{int}\{\psi = 0\}$, then

$$g'(\hat{t}) \leq 0.$$

(iii) If $\varphi(x,t) = f(x) + g(t)$, where $f, g \in C^1(\mathbb{R})$, and $u - \varphi$ has a global maximum at $(\hat{x}, \hat{t})$, and $\partial_{x_1} f(\hat{x}) \neq 0 \neq \partial_{x_2} f(\hat{x})$, then

$$g'(\hat{t}) \leq 0.$$

A viscosity supersolution is defined analogously by replacing upper semi-continuous, maximum and $\leq$ by lower semi-continuous, minimum and $\geq$, respectively.

A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Intuitively, $u - \varphi$ having a maximum at $(\hat{x}, \hat{t})$ can be interpreted as the graph of φ touching the graph of u from above. In the language of the level sets, the sub-level set of φ touches the boundary of the sub-level set of u from inside. Note that in two dimensions we do not need to consider $\operatorname{ess\,inf}$ or $\operatorname{ess\,sup}$ in tests (i_1) and (i_2) as written in [13, 14] because $\Lambda[\psi]$ is constant on each connected component of $\{\psi = 0\}$; see also [10, 12].

Theorem 3.4. *The function u defined in* (3.3) *and given explicitly in* (3.4) *is the viscosity solution of the crystalline curvature flow with initial data u_0.*

Proof. Let us show that u is a viscosity subsolution since this is more difficult. Indeed, u is nondecreasing in time and hence tests (ii) and (iii) are automatic for a supersolution.

Let us get the simpler tests (ii) and (iii) out of the way first.

Test (ii). Suppose that φ, ψ, g and $(\hat{x}, \hat{t})$ are as in Definition 3.3(ii). We need to show that $g'(\hat{t}) \leq 0$. Suppose therefore that $g'(\hat{t}) > 0$. We can find $\delta > 0$ such that $\psi(x) = 0 = \psi(\hat{x})$ for $\|x - \hat{x}\|_\infty < \delta$ and $g(t) < g(\hat{t})$ for $t \in (\hat{t} - \delta, \hat{t})$.

Let us set $\hat{\theta} := u(\hat{x}, \hat{t})$. By the maximality of $u - \varphi$ at $(\hat{x}, \hat{t})$, we get

$$u(x,t) \leq \hat{\theta} - \varphi(\hat{x}, \hat{t}) + \varphi(x,t) = \hat{\theta} + g(t) - g(\hat{t}) < \hat{\theta},$$

for $\|x - \hat{x}\|_\infty < \delta$, $t \in (\hat{t} - \delta, \hat{t})$. This can be interpreted in the following way: for every $t \in (\hat{t} - \delta, \hat{t})$ and any $\theta \in (\hat{\theta} + g(t) - g(\hat{t}), \hat{\theta}) \cap \Theta \neq \emptyset$ we have $x \in E_t^\theta$ but $x \notin E_{\hat{t}}^\theta$ for $\|x - \hat{x}\|_\infty < \delta$. But taking $t > \hat{t} - \frac{\delta^2}{2}$, we reach a contradiction with Lemma 3.2 that states that the neighborhood of $\hat{x}$ cannot disappear from E_t^θ that fast. Therefore $g'(\hat{t}) \leq 0$.

Test (iii). Suppose now that φ, f and g is as in Definition 3.3(iii). There is $\varepsilon > 0$ such that $\{x : f(x) = f(\hat{x}),\ |x - \hat{x}| < \varepsilon\}$ is a C^1 curve with normal vector $\hat{p} := \nabla f(\hat{x}) \neq 0$ at $\hat{x}$. Furthermore, as $u(\cdot, \hat{t}) - f$ has a global maximum at $\hat{x}$, we conclude that

$$u(x, \hat{t}) \leq u(\hat{x}, \hat{t}) + f(x) - f(\hat{x}) \qquad \text{in } \mathbb{R}^2,$$

and so

$$\{x : f(x) < f(\hat{x}),\ |x - \hat{x}| < \varepsilon\} \subset \{u(\cdot, \hat{t}) < \hat{\theta}\} =: U_{\hat{t}}^{\hat{\theta}}, \tag{3.6}$$

and $\hat{x} \in \partial U_{\hat{t}}^{\hat{\theta}}$. We can express the sub-level sets as

$$U_{\hat{t}}^{\hat{\theta}} := \bigcup_{\theta \in \Theta,\ \theta < \hat{\theta}} E_{\hat{t}}^{\theta}.$$

See Figure 3 for the boundaries of $U_t^{\hat{\theta}}$ for $0 < t < t^{*,\hat{\theta}}$.

In the following, we say that an edge of $A \subset \mathbb{R}^2$, that is, a maximal line segment Γ of ∂A, is *convex* if A is convex in its neighborhood ( ). More precisely, there exists a convex neighborhood C of the edge Γ such that $A \cap C$ is convex. Similarly, the edge Γ is *concave* if $A^{\mathsf{c}} \cap C$ is convex for some convex neighborhood C ( ). Finally, if the edge is neither, we say that it is *indeterminate* ( ). Similarly we will talk about convex ( ) or concave ( ) corners.

Therefore by (3.6) and the fact that $\hat{p}$ is not parallel to either of the axes, $\hat{x}$ can be located only on the concave corners of the boundary of $U_{\hat{t}}^{\hat{\theta}}$. More precisely, $\hat{\theta} \in (0, 1]$, $\hat{x}_1 = -\hat{x}_2$ and $0 < \hat{t} < t^{\sharp,\hat{\theta}} = 1 + 2\hat{\theta}$. But then $u(\hat{x}, t) = u(\hat{x}, \hat{t})$ for t near $\hat{t}$ because this corner does not move and hence $g'(\hat{t}) = 0$.

Test (i$_1$). It is time to tackle the more challenging test (i$_1$). We will see that here we test the horizontal facets, that is, the one dimensional edges parallel to the x_1-axis. Suppose that φ, ψ, f, g and $(\hat{x}, \hat{t})$ are as in Definition 3.3(i$_1$). We need to estimate the quantity $\Lambda[\psi](\hat{x}_1)$ from below, which means controlling the length of the component of $\{\psi = 0\}$ containing $\hat{x}_1$ and the behavior of ψ near its boundary points.

We again set $\hat{\theta} := u(\hat{x}, \hat{t})$. The maximality of $u - \varphi$ at $(\hat{x}, \hat{t})$ is equivalent to

$$\psi(\hat{x}_1 + v_1) + f(\hat{x}_2 + v_2) + g(\hat{t}) = \varphi(\hat{x} + v, \hat{t}) \geq u(\hat{x} + v, \hat{t}) - \hat{\theta} + \varphi(\hat{x}, \hat{t}),$$

for $v \in \mathbb{R}^2$, which after rearrangement using $\psi(\hat{x}_1) = 0$ reads

$$\psi(\hat{x}_1 + v_1) \geq u(\hat{x} + v) - \hat{\theta} + f(\hat{x}_2) - f(\hat{x}_2 + v_2). \tag{3.7}$$

We immediately have

$$\psi(x_1) \geq 0 \qquad\qquad \text{whenever } (x_1, \hat{x}_2) \notin U_{\hat{t}}^{\hat{\theta}} := \{u(\cdot, \hat{t}) < \hat{\theta}\}. \tag{3.8}$$

Let $s = f'(\hat{x}_2)/|f'(\hat{x}_2)|$ be the sign of $f'(\hat{x}_2) \neq 0$. There is $\delta > 0$ such that $f(\hat{x}_2) - f(\hat{x}_2 - hs) > 0$ for $h \in (0, \delta)$. Therefore by (3.7) we get

$$\psi > 0 \quad \text{on } \{x_1 : (x_1, \hat{x}_2 - hs) \notin U_{\hat{t}}^{\hat{\theta}} \text{ for some } h \in (0, \delta)\}. \tag{3.9}$$

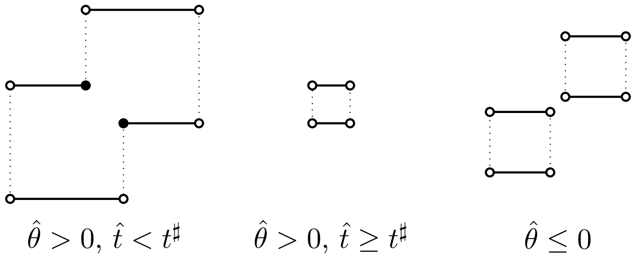

Figure 3: Possible locations of $\hat{x}$ on the boundary of $U_{\hat t}^{\hat\theta}$ for test (i$_1$) denoted by thick line segments.

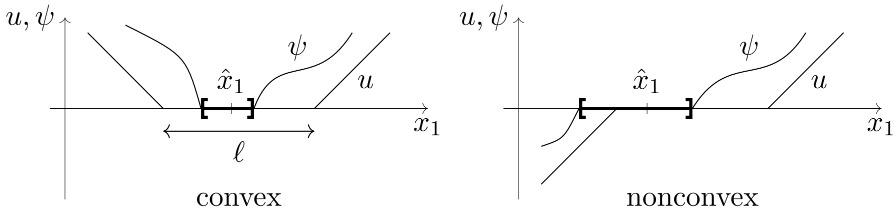

Figure 4: The graphs of ψ (above) and $u((\cdot, \hat{x}_2), \hat{t}) - \hat{\theta}$ on a convex edge (left) vs. nonconvex edge (right). The set $\{\psi = 0\}$ is denoted by the bracketed interval.

On the other hand, by the assumption on the test function, we have $\psi(\hat{x}_1 + h, \hat{t}) = 0$ for $|h| < \varepsilon$ for some $\varepsilon > 0$ and therefore (3.9) yields

$$\{\hat{x} + v : |v_1| < \varepsilon,\ -\delta < s v_2 < 0\} \subset U_{\hat t}^{\hat\theta}.$$

This implies that $\hat{x}$ must lie on one of the horizontal edges of the boundary of $U_{\hat t}^{\hat\theta}$ away from the convex corners; see Figure 3.

If $\hat{x}$ is on a convex edge, both boundary points of the component of $\{\psi = 0\}$ containing $\hat{x}$ are positive due to (3.8) and (3.9), and hence $\chi(\hat{x}) = 2$; see Figure 4(left). Furthermore, the length of the component is smaller than the length ℓ of the edge by (3.9). We conclude that $\Lambda[\psi](\hat{x}) \geq \frac{2}{\ell}$.

On the other hand, if $\hat{x}$ does not lie on a convex edge, we can only conclude that at least one boundary point of the component of $\{\psi = 0\}$ has a positive sign and therefore $\chi(\hat{x}) \geq 0$; see Figure 4(right). This gives us only that $\Lambda[\psi](\hat{x}) \geq 0$.

To estimate the time derivative $g'(\hat{t})$, we choose $\xi \in C^1(I, \mathbb{R}^2)$ and I a neighborhood of $\hat{t}$ such that $\xi(t) \in \partial U_t^{\hat\theta}$ for $t \in I$. Then

$$u(\xi(t), t) - \varphi(\xi(t), t) \leq u(\hat{x}, \hat{t}) - \varphi(\hat{x}, \hat{t}) = \hat{\theta} - \varphi(\xi(\hat{t}), \hat{t}).$$

But $u(\xi(t), t) = \hat{\theta}$ and therefore $t \mapsto \varphi(\xi(t), t)$ has a minimum at $\hat{t}$. In particular,

$$0 = \tfrac{d}{dt} \varphi(\xi(t), t)|_{t = \hat{t}} = g'(\hat{t}) + f'(\hat{x}_2)\xi_2'(\hat{t}).$$

Now $s\xi_2'(\hat{t})$ just happens to be the outer normal velocity of the horizontal edge of $U_{\hat t}^{\hat\theta}$ on which $\hat{x}$ lies. So if $\hat{x}$ lies on a nonconvex edge, we can simply choose $\xi(t) = \hat{x}$ since such edges do not move by construction, and so we have

$$g'(\hat{t}) = 0 \leq |f'(\hat{x}_2)| \Lambda[\psi](\hat{x}_1).$$

On the other hand, on the convex edges, we have by the crystalline algorithm $s\xi'(\hat{t}) = -\frac{2}{\ell}$, where ℓ is the length of the edge. Therefore we conclude that if $\hat{x}$ is on the convex edge,

$$g'(\hat{t}) = -|f'(\hat{x}_2)|s\xi'(\hat{t}) = |f'(\hat{x}_2)|\frac{2}{\ell} \leq |f'(\hat{x}_2)|\Lambda[\psi](\hat{x}_1).$$

Test (i$_2$). This test can be verified as test (i$_1$) by symmetry.

This concludes the proof that u defined in (3.3) and given by the explicit formula (3.4) is a viscosity subsolution of the level set equation (2.3) in the sense of Definition 3.3. The proof that it is a supersolution is similar in (i$_1$) and (i$_2$), and trivial in cases (ii) and (iii) as u is nondecreasing in time. □

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Faculty of Mathematics and Physics
Institute of Science and Engineering
Kanazawa University
Kakuma town, Kanazawa, Ishikawa 920-1192
JAPAN
E-mail address: `npozar@se.kanazawa-u.ac.jp`