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DIPLOMOVÁ PRÁCE

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Vybrané vlastnosti stacionárních axiálně symetrických polí v obecné relativitě

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Abstrakt: *Tato práce je věnována použití metod algebraické geometrie při hledání stacionárních axiálně symetrických prostoročasů jako přesných řešení Einsteinových rovnic obecné teorie relativity. Úloha se dá převést na Ernstovu rovnici – úplně integrabilní nelineární parciální diferenciální rovnici druhého řádu. Ačkoli je známa řada konkrétních řešení této rovnice popisujících gravitační pole izolovaných objektů (černých děr, prstencových a diskových zdrojů) i široké třídy velmi obecných řešení získané tzv. generačními technikami, není zatím plně zvládnuta okrajová úloha pro dostatečně obecnou situaci, např. pro případ vícenásobných zdrojů (zejména černých děr s disky či prstenci), který je nejen teoreticky zajímavý, ale i astrofyzikálně významný. Jako slibné se ukazuje převedení Ernstovy rovnice na odpovídající lineární úlohu, její formulace jako Riemannova-Hilbertova problému a řešení pomocí theta funkcí na Riemannových plochách. Po stručném přehledu fyzikální motivace a nezbytných matematických partií uvádíme hlavní výsledky nedávných prací Kleina a Korotkina (et al.), přičemž upřesňujeme nebo více rozvádíme některé body jejich postupu. Navrhujeme poněkud odlišný důkaz dvou klíčových vět a také 'rychlý' numerický kód pro vyčíslení řešení Ernstovy rovnice ve tvaru theta-funkcí (podobný program mezitím uveřejnili Klein a Frauendiener).*
Klíčová slova: *přesná řešení Einsteinových rovnic, Ernstova rovnice, metoda inverzního rozptylu, algebraická geometrie, Riemannův-Hilbertův problém*

Title: *Selected properties of stationary axially symmetric fields in general relativity*
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Abstract: *This thesis is devoted to the usage of the methods of algebraic geometry in finding stationary axially symmetric spacetimes as exact solutions to Einstein equations of general relativity. The task can be transformed into Ernst equation – completely integrable second-order non-linear partial differential equation. Although a number of particular solutions of this equation are known describing gravitational field of isolated objects (black holes, ring or disc sources), as well as wide classes of very general solutions obtained by the so called generation techniques, there remains not fully managed the boundary-value problem for sufficiently generic situation, e.g. for multiple sources (mainly black holes with discs or rings) which is both theoretically interesting and astrophysically important. It is promising to translate the Ernst equation into the corresponding linear task, to formulate this as Riemann-Hilbert problem and to solve the latter in terms of theta functions on Riemann surfaces. After a brief survey of physical motivation and of necessary mathematical parts, we give main results of the recent works by Klein and Korotkin (et al.), trying to make them more precise and/or detailed in certain points. We suggest a somewhat different proof of two key theorems and also a 'fast' numerical code for evaluation of the Ernst-equation solution in terms of theta-functions (in the meantime, a similar program has been published by Klein and Frauendiener).*
Keywords: *exact solutions of Einstein equations, Ernst equation, inverse scattering method, algebraic geometry, Riemann-Hilbert problem*

1 Introduction

The first theory of gravitation was developed by Isaac Newton in 17th century. Since then it was successfully used to describe and explain the behavior of the Solar system. But when the special theory of relativity appeared at the beginning of the 20th century, problems arose, in particular with superluminal (infinite) velocity of the gravitational force. A brand new point of view was brought by the general theory of relativity (general relativity) published in 1915 by Albert Einstein. It described gravitation as a manifestation of a curvature of the spacetime itself. It was in accordance with special relativity, but it called for a completely new model of our universe. Since then many other theories of gravitation have been developed but general relativity is still assumed to be the correct one in a wide range of situations, although not all of its predictions have been confirmed by experiments yet.

The fundation-stone of general relativity are Einstein equations. They describe the interaction between physical fields and bodies and the spacetime. But they are much more involved then the simple equation for the gravitational force in the Newton theory. The Einstein equations are non-linear because the gravitation interaction itself has energy. The equations are so complex that even Einstein thought that no any exact solution would be ever found. But in 1916 Karl Schwarzschild published his famous exact solution in a special case. Since then many methods to solve the Einstein equations were developed. One can find exact or numerical solutions, or solutions to linearized or perturbed systems. In this work, we are interested in exact solutions to Einstein equations in a stationary axisymmetric vacuum spacetime. In this case, the Einstein equations can be reduced to one partial differential equation (Ernst equation) for one complex function known as Ernst potential. The most interesting and useful classes of known solutions (Kerr) belong here. However, they were found "ad-hoc". We are interested in more systematic approach.

Exact solutions of physical equations are important for theoretical reasons as well as in understanding and modelling specific natural phenomena. Although there are approximate (perturbative) and numerical techniques, the outcomes of the former are typically valid within a narrow range of parameters only, while those of the latter go short of generality: not sure whether a given numerical result represents a typical or just a marginal case, one can hardly recognise, analyse and interpret different *classes* of solutions. Also — from a purely mathematical view — the richness embodied in the equations can only be fully revealed on their exact solutions.

During 1960s, new powerful methods ("inverse scattering" methods) were developed to generate exact solutions of non-linear partial differential equations, based on their groups of symmetries. Such symmetries imply the existence of a certain linear system which is equivalent to the original non-linear system in the sense that the latter is its integrability condition. The problem is thus translated into the solution of a linear system for a matrix function which is dependent on an additional ("spectral") parameter. The methods were first successfully applied to completely integrable non-linear evolution (hyperbolic) PDEs, and later to elliptic PDEs like the Ernst equation, too.

The Ernst equation is a non-linear second-order system of two equations for the real and imaginary parts of the Ernst potential. It is completely integrable in the Hamiltonian sense: the equations yield the same number of conserved quantities as degrees of freedom [18]. The symmetry group is the infinite dimensional Geroch group [2].

The study of the Ernst equation by means of Bäcklund transformations and solitonic methods has brought a lot of new solutions [13, 28, 30, 32–34, 38]. They are given in terms of algebraic or exponential functions and can involve an infinite number of parameters [31]. However, with the increasing number of parameters, the solutions become pretty cumbersome and only a very few of them have been interpreted in detail up to now. It is probable that almost all of them do not correspond to any physical situation. Namely, within these approaches it is very problematic to set certain physical requirements on the solution in advance, so they yield rather "pot shot" results.

In the 1980s, a new rich class of solutions was found by the methods of algebraic geometry. Loosely speaking, it consists in posing the linear problem as a Riemann-Hilbert problem which can be solved in terms of theta functions living on certain Riemann surfaces. The approach, already considered for several other physical equations before, was employed for the Ernst equation by Korotkin [23, 24, 26, 27] and later worked out by Klein and Richter [10, 15, 16, 20, 21]. The papers from Klein and Richter were published as a book [22]. The solutions have many promising features and contain physically interesting subclasses which might hopefully describe fields of realistic sources. However, they have not attracted much attention. One of the likely reasons is the non-trivial theory needed for their formulation, which is still being developed [19, 22]. Even a code for their numerical evaluation is difficult to construct except of rather special cases [10–12]. One of the purposes of the present work is to address this problem.

In section 2, we shortly introduce general relativity and the Einstein equations, we derive the Ernst equation in detail and we also briefly discuss the electro-vacuum case based on the papers by Ernst [7, 8]. In sections 3 and 4, we present the theory of Riemann surfaces and theta functions based on the book Farkas & Kra [9] that is necessary for algebro-geometric methods studied later. In section 5, we summarize the results on Riemann-Hilbert problems that we will use later. Section 6 illustrates the algebro-geometric methods for solving nonlinear evolution systems, it is based on the book by Belokolos et al. [1].

We present our own results in sections 7 and 8. We focused on algebro-geometric methods for construction of the Ernst equation. We investigate two versions of the linear system used in literature. We prove two theorems that summarize the properties of the matrix functions that solve these systems. Our proofs are more detailed and somewhat more precise then those found in papers [18, 21, 27]. We also give a very detailed proof that the function constructed in [21] using a scalar Riemann-Hilbert problem is really a solution to the Ernst equation. The relevance of hyperelliptic Riemann surfaces is discussed in a great detail.

The numerical evaluation is rather challenging. We present an implementation of a code that is powerful enough to evaluate quantities of interest on hyperelliptic Riemann surfaces in section 8. Even though the code is far from complete, we obtained some results that are comparable to the result in [11, 12].

2 General relativity

General relativity is the geometrical theory of gravitation published by Albert Einstein in 1915. It unifies special relativity and Isaac Newton's law of universal gravitation with the insight that gravitation is not viewed as being due to a force (in the traditional sense) but rather a manifestation of curved space and time, this curvature being produced by the mass-energy content of the spacetime.

In this section, we briefly introduce Einstein equations as a complete set of equations governing the interaction between bodies or physical fields and geometry of spacetime. We show how these equations can be simplified in the case of stationary axisymmetric fields. At the end we discuss some simple examples of solutions to these equations. For details, see some of the standard book of general relativity, for example Wald [38]. For a more mathematical point of view, see Hawking & Ellis [14].

2.1 Einstein equations

The main result of general relativity are Einstein equations, a system of 10 partial differential equations describing interaction between mass and energy and geometry of spacetime. The equations can be written in the standard form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor and $T_{\mu\nu}$ is the stress-energy tensor. The constants G, c and Λ are the physical constants in the equations. We will use metrized units, where the gravitational constant G and the speed of light c are set to be 1. We will assume that the cosmological constant Λ is zero.

The Ricci tensor and Ricci scalar are obtained from the Riemann curvature tensor that measures a local curvature of the spacetime. On the other hand, the stress-energy tensor measures the amount and distribution of energy and mass in the spacetime.

Since the Bianchi identities hold for the Riemann curvature tensor, only 6 of the Einstein equations are equations for the metric functions. The other 4 are the law of conservation of energy and momentum (which is a very interesting and important feature of these equations). Even though we have only 6 equations for 10 unknown metric functions, we have enough equations to found the complete metric. The metric tensor has to be independent of the choice of coordinates, i. e. up to a diffeomorphism given by 4 functions, and thus by fixing a coordinate system we fix 4 metric functions. This dependence on a coordinate system causes problems when solving the equations globally, because particular coordinates are good only locally and one needs to continue the solution to other coordinate systems to obtain a solution on the whole spacetime. It is also difficult to recognize positions of singularities of the spacetime or to see that two different solutions describe the same spacetime.

The Einstein equations comprise a hyperbolic system of 6 nonlinear partial differential equations for 6 independent functions. The initial value problem is well-posed with initial conditions given on a 3-dimensional submanifold. The nonlinearity of the equations and the dependence on coordinates, however, make analytic or numerical solution difficult. Some restrictions on the metric functions and on the coordinates are usually imposed. In this work, we are interested in a stationary axially symmetric case in vacuum.

2.2 Derivation from the action principle

Einstein equations can be derived using an action principle (Carroll [3]). The action (so called Hilbert action) is the integral of the simplest Lagrangian density over spacetime

$$S_{\mathrm{H}} = \int \mathcal{L}_{\mathrm{H}} d^4x. \tag{2.1}$$

The Lagrange density is a tensor density, which can be expressed as $\sqrt{-g}$ times a scalar. Any nontrivial scalar must contain at least second derivatives of the metric. The only independent scalar the contains no higher then second derivatives of the metric and it depends on the second derivatives linearly is the Ricci scalar R. The simplest possible choice of the Lagrangian density as proposed by Hilbert for a vacuum spacetime is therefore

$$\mathcal{L}_{\mathrm{H}} = \sqrt{-g}R. \tag{2.2}$$

Varying the action S_{H} with respect to metric $g_{\mu\nu}$ gives the equation of motion, the Einstein equations in vacuum. To obtain the full set of the Einstein equation, the action has to be modified slightly.

2.3 Stationary axially symmetric field

The stationary axially symmetric solutions to the Einstein equations are astrophysically important since they can describe the gravitational fields of rotating bodies in equilibrium. It is generally believed that most of the stars and galaxies can be described in a good approximation as stationary axially symmetric fluid bodies. In addition, they are usually reflectionally symmetric with respect to some "equatorial" plane. A relativistic treatment is necessary in particular for very compact objects such as neutron stars and black holes, whose extremely strong fields are not described properly by Newton's theory of gravitation.

The stationary axially symmetric spacetime is characterized by two commuting Killing fields[1], the timelike[2] Killing field ∂_t and the spacelike Killing field ∂_ϕ with closed orbits. Every vacuum stationary axisymmetric metric can always be cast into the Weyl-Lewis-Papapetrou form

$$\mathrm{d}s^2 = -e^{2U}(\mathrm{d}t + a\,\mathrm{d}\phi)^2 + e^{-2U}(e^{2k}(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2) + \rho^2\mathrm{d}\phi^2), \qquad (2.3)$$

where t and ϕ are coordinates adapted to Killing fields and ρ, ζ span the polar sections. The functions U, k and a are functions of Weyl coordinates ρ and ζ only. The metric function U reduces to gravitational potential in the Newtonian limit. a is the "gravitomagnetic" potential. In the following, we will use a new function $f = e^{2U}$.

2.4 Ernst equation

Ernst showed (see [7]) that the vacuum field equations in the stationary axisymmetric case are equivalent to the Ernst equation for the complex potential $\mathcal{E} = \mathcal{E}(\rho, \zeta)$

$$(\Re\mathcal{E})\Delta\mathcal{E} = \nabla\mathcal{E}.\nabla\mathcal{E}, \qquad (2.4)$$

where Δ is the Laplace operator in cylindrical coordinates (ρ, ζ, ϕ) $\Delta = \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{\zeta\zeta}$, and ∇ is a gradient in cylindrical coordinates $\nabla = (\partial_\rho, \partial_\zeta, 0)$ (∂_ϕ is omitted due to axisymmetry).

The Ernst equation can be derived using tensor methods from Einstein equations, but the simpler way is to use the Lagrangian density[3] (Ernst [7])

$$\mathcal{L} = -\frac{1}{2}\rho f^{-2}\nabla f \cdot \nabla f + \frac{1}{2}\rho^{-1}f^2\nabla a \cdot \nabla a.$$

Varying[4] the functions f and a we obtain the field equations

$$f\Delta f = \nabla f \cdot \nabla f - \rho^{-2}f^4\nabla a \cdot \nabla a, \qquad (2.5)$$

$$\nabla \cdot (\rho^{-2}f^2\nabla a) = 0. \qquad (2.6)$$

[1] A Killing field is a vector field such that the Lie derivative of the metric along this field is zero.

[2] Negative norm.

[3] The terms with e^{2k} are removed since they do not change the resulting formulas and the metric function k can be expressed in terms of the other metric functions.

[4] We compute the Gâteaux derivative $\frac{d}{dt}\int_\Omega \mathcal{L}(f + th)\rho d\rho d\zeta\big|_{t=0}$ in f of the functional $\int_\Omega \mathcal{L}$ in any direction h such that $h \to 0$ as $\rho \to \infty$ or $\zeta \to \pm\infty$. Then we can apply the Green's theorem and we get (2.5). When we compute the Gâteaux derivative for a instead, we get (2.6).

If n is a unit vector in the azimuthal direction and φ is any reasonable function independent of azimuth, we have the identity[5]

$$\nabla \cdot (\rho^{-1} n \times \nabla\varphi) = 0. \tag{2.7}$$

Now equation (2.6) may be regarded[6] as the integrability condition for the existence of the function $b = b(\rho, \zeta)$ independent of azimuth defined by

$$\rho^{-1} f^2 \nabla a = n \times \nabla b. \tag{2.8}$$

This relation is equivalent to[7]

$$f^{-2} \nabla b = -\rho^{-1} n \times \nabla a, \tag{2.9}$$

therefore the identity (2.7) implies[8] the field equation

$$\nabla \cdot (f^{-2} \nabla b) = 0 \tag{2.10}$$

for the new potential b. When we express equation (2.5) in terms of the function φ and compare it with equation (2.10), we see that the complex function

$$\mathcal{E} = f + \mathrm{i}b \tag{2.11}$$

satisfies the simple differential equation (2.4).

In the Weyl-Lewis-Papapetrou coordinate system (2.3), the Ernst equation has the form

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{\zeta\zeta} + \frac{1}{\rho}\mathcal{E}_\rho + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_\zeta^2 + \mathcal{E}_\rho^2), \tag{2.12}$$

where the bar denotes complex conjugation in $\mathbb{C}$. The metric functions U, k and a can be expressed in terms of the Ernst potential $\mathcal{E}$ as ([18])

$$e^{2U} = f = \Re\mathcal{E}, \quad a_\xi = 2\rho\frac{(\mathcal{E} - \bar{\mathcal{E}})_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2}, \quad k_\xi = (\xi - \bar{\xi})\frac{\mathcal{E}_\xi \bar{\mathcal{E}}_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2}, \tag{2.13}$$

where ξ is a new complex coordinate $\xi = \zeta - \mathrm{i}\rho$.

Since $\partial_\xi = \frac{1}{2}(\partial_\zeta + \mathrm{i}\partial_\rho)$ and $\partial_{\bar{\xi}} = \frac{1}{2}(\partial_\zeta - \mathrm{i}\partial_\rho)$, the Ernst equation can be rewritten using the coordinate ξ in an equivalent form

$$\mathcal{E}_{\xi\bar{\xi}} - \frac{1}{2(\xi - \bar{\xi})}(\mathcal{E}_\xi - \mathcal{E}_{\bar{\xi}}) = \frac{2}{\mathcal{E} + \bar{\mathcal{E}}}\mathcal{E}_\xi \mathcal{E}_{\bar{\xi}}. \tag{2.14}$$

The metric function U follows directly from a solution $\mathcal{E}$ by definition of the Ernst potential. a and k can be obtained from (2.13) via quadratures. The Ernst equation (2.14) is the integrability condition for (2.13).

[5]Use the product rule and observe that $\nabla \times n = -\frac{\nabla\rho}{\rho} \times n + \frac{2}{\rho}\nabla\zeta$ and $\nabla\zeta = \nabla\rho \times n$. This implies $\nabla\varphi \cdot (\frac{\nabla\rho}{\rho} \times n) = \nabla\varphi \cdot (\nabla \times n)$.

[6](2.6) implies (2.7)

[7]Apply $n\times$ to both sides.

[8]Apply $\nabla\cdot$ to both sides of (2.9) and use (2.7) for $\varphi := a$.

2.5 Examples of Ernst potentials

Here we will present Ernst potentials for simple well-known solutions.

2.5.1 Minkowski solution

The Minkowski metric is the metric of flat spacetime. The corresponding Ernst potential is constant, $\mathcal{E} = 1$.

2.5.2 Kerr solution

The Kerr solution belongs to the class of stationary axisymmetric spacetimes and thus must be a solution of the Ernst equation. Neugebauer and Meinel showed how to derive this solution from the boundary data on the axis [34]. The Ernst potential can be expressed in the form [17]

$$\mathcal{E} = \frac{e^{-\mathrm{i}\varphi} r_+ + e^{\mathrm{i}\varphi} r_- - 2m\cos\varphi}{e^{-\mathrm{i}\varphi} r_+ + e^{\mathrm{i}\varphi} r_- + 2m\cos\varphi},$$

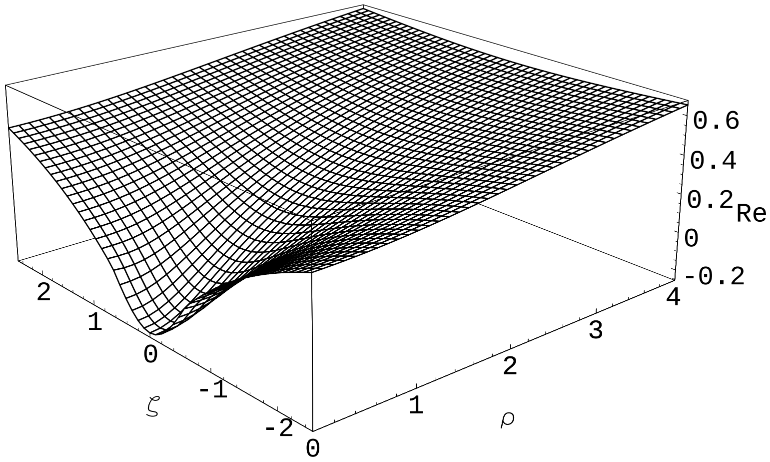
where $r_\pm = \sqrt{(\zeta \pm m\cos\varphi)^2 + \rho^2}$. The solution is parametrized by two parameters m and φ. They can be related to the Arnowitt-Deser-Misner (ADM) mass m and the angular momentum $J = m^2 \sin\varphi$. The black hole has a horizon located on the axis $\rho = 0$ in the interval $\zeta \in [-m\cos\varphi, m\cos\varphi]$. For $\varphi = 0$ we get the Schwarzschild solution and for $\varphi = \pi/2$ the solution becomes the extreme Kerr solution.

2.6 Einstein-Maxwell equations in stationary axisymmetric case

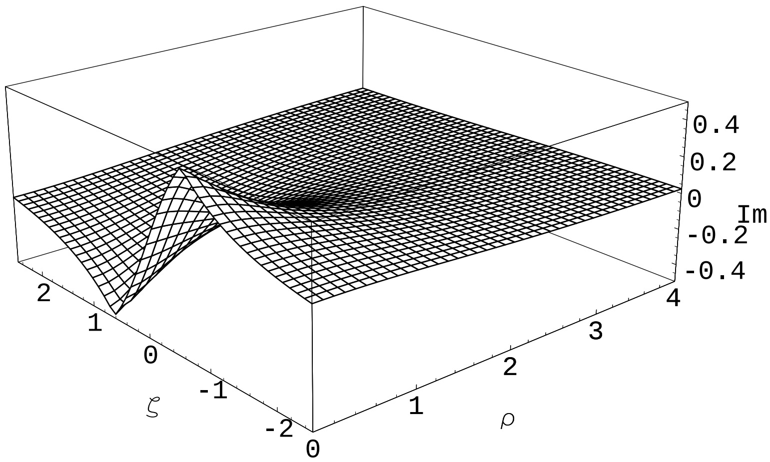
In this section, we release the condition that the spacetime is empty and allow it to contain an electromagnetic field. We can simplify the Einstein equations (Einstein-Maxwell equations in this case) in a way similar to Section 2.4 as was first noted by Ernst [8]. We can write the metric in the form (2.3) again. In this case, the Lagrangian density is

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}\rho f^{-2}\nabla f \cdot \nabla f + \frac{1}{2}\rho^{-1} f^2 \nabla a \cdot \nabla a + 2\rho f^{-1} A_t \nabla A_t \cdot \nabla A_t \\ - 2\rho^{-1} f(\nabla A_\phi + a\nabla A_t)\cdot(\nabla A_\phi + a\nabla A_t),\end{aligned}$$

where A_ϕ and A_t are the ϕ and t components of the electromagnetic 4-potential, respectively. By variation of the 4 functions f, a, A_ϕ and A_t, we



(a) $\Re\mathcal{E}$



(b) $\Im\mathcal{E}$

Figure 1: The Ernst potential of the Kerr solution with parameters $m = 0.5$ and $\varphi = 0.9$.

can derive the field equations in the form

$$\begin{aligned} (\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi})\Delta\mathcal{E} &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\mathcal{E}, \\ (\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi})\Delta\Phi &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\Phi, \end{aligned} \tag{2.15}$$

where the new complex functions $\mathcal{E}$ and Φ are defined by

$$\begin{aligned} \mathcal{E} &= (e^{2U} - \Phi\bar{\Phi}) + \mathrm{i}b', \\ \Phi &= A_t + \mathrm{i}A'_\phi, \end{aligned}$$

where b' and A'_ϕ are new potentials independent of azimuth, defined by

$$\begin{aligned} n \times \nabla A'_\phi &= \rho^{-1} f(\nabla A_\phi + a\nabla A_t), \\ n \times \nabla b' &= -\rho^{-1} f^2 \nabla a - 2n \times \mathrm{Im}(\overline{\Phi}\nabla\Phi). \end{aligned}$$

n is a unit vector in azimuthal direction. For details of the derivation, see [8] or [18]. The Ernst equation (2.4) can be obtained from (2.15) under the assumption $\Phi = 0$.

3 Riemann surfaces

The theory of Riemann surfaces appear in many areas of mathematics. In this thesis we will use it as a tool for the construction of exact solutions to the Ernst equation. Riemann surfaces are studied in detail for example in Farkas & Kra [9], we give just a short account here. We will be especially interested in compact hyperelliptic Riemann surfaces, because they will appear in the solutions.

Definition 3.1. *A* Riemann surface *is a connected one-dimensional complex analytic manifold M with a maximal set of charts $\{U_\alpha, z_\alpha\}$, $\alpha \in A$, on M such that the* transition functions

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta) \tag{3.1}$$

are holomorphic whenever $(U_\alpha \cap U_\beta) \neq \emptyset$.

The simplest Riemann surfaces are the complex plane $\mathbb{C}$, the one point compactification, $\mathbb{CP}^1 \equiv \mathbb{C} \cup \{\infty\}$, of $\mathbb{C}$ (known as the *extended plane* or *Riemann sphere*) or any domain (connected open subset) on any Riemann surface.

Definition 3.2. *A compact Riemann surface is called* closed*, a non-compact surface is called* open*.*

We do not need to specify a maximal set of analytic coordinate charts, merely a cover by any set of analytic coordinate charts (see [9], p. 10).

Definition 3.3 (Holomorphic mapping)**.** *A continuous mapping*

$$f : M \to N \tag{3.2}$$

between Riemann surfaces is called holomorphic *(or* analytic*) if for every local coordinate $\{U, z\}$ on M and every local coordinate $\{V, \zeta\}$ on N with $U \cap f^{-1}(V) \neq \emptyset$, the mapping*

$$\zeta \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \to \zeta(V) \tag{3.3}$$

is holomorphic (as a mapping from $\mathbb{C}$ to $\mathbb{C}$).

Definition 3.4. *A holomorphic mapping into $\mathbb{C}$ is called a* holomorphic function*. A holomorphic mapping into $\mathbb{C} \cup \{\infty\}$, other then mapping sending all points of M to ∞, is called a* meromorphic function*. The vector space of all meromorphic functions on a Riemann surface M is denoted $\mathcal{K}(M)$. The mapping $f : M \to N$ is called* constant *if $f(M)$ is a point.*

A holomorphic mapping is open[9]. It can be shown that if $f : M \to N$ is a holomorphic mapping and M is compact, then f is either constant or surjective. In the latter case, N is also compact. In particular, the only holomorphic functions on a compact Riemann surface are constants, because $\mathbb{C}$ is not compact. On the other hand, there are non-constant meromorphic functions since the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is compact.

We now want to define the multiplicity of a holomorphic mapping f. WConsider a non-constant holomorphic mapping between Riemann surfaces $f : M \to N$. For a given point $P \in M$ we choose local coordinates $\tilde{z}$ on M vanishing at P and ζ vanishing at $f(P)$. In terms of these local coordinates, we can write

$$\zeta = f(\tilde{z}) = \sum_{z \geq n} a_k \tilde{z}^k, \quad n > 0, a_n \neq 0.$$

We also have

$$\zeta = \tilde{z}^n h(\tilde{z}^n) = (\tilde{z} h(\tilde{z}))^n,$$

where h is holomorphic and $h(0) \neq 0$. $\tilde{z}h(\tilde{z})$ is another local coordinate vanishing at P, and in terms of this new coordinate the mapping f is given by

$$\zeta = z^n. \tag{3.4}$$

Definition 3.5. *The number n defined above is the* ramification number of f at P. *We say that* f takes on the value $f(P)$ n-times at P *or* f has multiplicity n at P. *The number* $(n-1)$ *is called the* branch number of f at P *and denoted* $b_f(P)$.

The following theorem can be proved:

Theorem 3.1. *Let $f : M \to N$ be a non-constant holomorphic mapping between compact Riemann surfaces. There exists a positive integer m such that every $Q \in N$ is assumed precisely m times on M by f, accounting for multiplicities. That is, for all $Q \in N$,*

$$\sum_{P \in f^{-1}(Q)} (b_f(P) + 1) = m.$$

Definition 3.6. *The number m above is called the* degree *of f and denoted* $\deg f$. *We say that f is an m*-sheeted cover of N by M.

[9] Open sets map to open sets.

3.1 Topology of Riemann surfaces

In this section, a *curve* on M is a continuous map c of the closed interval $I = [0, 1]$ into M. The point $c(0)$ is called the *initial point* of the curve and $c(1)$ is called the *terminal* or *end point* of the curve.

If P, Q are two points of M and c_1 and c_2 are two curves on M with initial point P and terminal point Q, we say that c_1 is *homotopic* to c_2 ($c_1 \sim c_2$) provided that there is a continuous map $h : I \times I \to M$ with the properties $h(t, 0) = c_1(t)$, $h(t, 1) = c_2(t)$, $h(0, u) = P$ and $h(1, u) = Q$ (for all $t, u \in I$). In other words, one curve can be continuously deformed into the other.

We now consider all closed curves on M which pass through any point P, of M i. e. all curves with initial and terminal point P. Curves c_1 and c_2 are equivalent whenever they are homotopic. The set of equivalence classes of closed curves through P forms an Abelian group. The sum of the equivalence class of the curve c_1 with the equivalence class of the curve c_2 is the equivalence class of the curve c_1 followed by c_2. The inverse of the equivalence class of the curve c is the curve c parametrized in the reverse direction. The group of equivalence classes constructed in this way is called the *fundamental group of M based at P*.

The fundamental group based at P and the fundamental group based at Q are almost canonically isomorphic as groups. The isomorphism between them depends only on the homotopy class[10] of the path from P to Q.

Definition 3.7. *The* fundamental group of *M, $\pi_1(M)$, is defined to be the fundamental group of M based at P, for any $P \in M$.*

For our applications, the dependence of $\pi_1(M)$ on the base point P will be irrelevant.

The fundamental group is isomorphic the *first simplicial homology group* $H_1(M)$ (see [9], p. 15).

For a compact Riemann surface M, it can be shown (e. g. [9]) that the fundamental group is either trivial (all closed curves are equivalent to a point) or generated by $2g$ closed curves $a_1, \ldots, a_g, b_1, \ldots, b_g$.

Definition 3.8 (Genus)**.** *In the former case we say that the* genus *of M is zero and in the latter case we say that the* genus *is g.*

Consider a non-constant holomorphic mapping $f : M \to N$ between compact Riemann surfaces. Assume that M is a surface of genus g, N is a surface of genus γ. Assume that f is of degree n. We define the *total*

[10]The equivalence class of homotopic curves.

branching number of f by

$$B = \sum_{P \in M} b_f(P).$$

Theorem 3.2 (Riemann-Hurwitz Relation)**.** *We have*

$$g = n(\gamma - 1) + 1 + B/2.$$

We will also need the following definition:

Definition 3.9 (Covering manifolds)**.** *The manifold M^* is said to be a* covering manifold *of the manifold M provided there is a continuous surjective map (called a* covering map*) $f : M^* \mapsto M$ with the following property: for each $P^* \in M^*$ there exist a local coordinate z^* on M^* vanishing at P^*, a local coordinate z on M vanishing at $f(P)$, and an integer $n > 0$ such that f is given by $z = z^{*n}$ in terms of these local coordinates. Here the integer n depends only on the point $P^* \in M^*$. If $n > 1$, P^* is called a* branch point of order $n - 1$ *or a* ramification point of order n.

3.2 Intersection theory on compact surfaces

A single non-negative integer, called *genus*, provides a complete topological classification of compact Riemann surfaces. Every compact Riemann surface is topologically equivalent to a sphere with g handles. The curves from the basis of the fundamental group $\pi_1(M)$ are denoted $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$. We want to construct a basis called the canonical basis of the fundamental group by curves that intersect in a specific way.

Let a and b be two closed curves on the Riemann surface M. We define the *intersection number of a and b*, $a \cdot b$, as the number of intersections of the type shown in Figure 2 minus the number of intersections of the type from Figure 2 when the curves a and b are interchanged. The intersection number is well defined and only depends on the homology classes of a and b.

Definition 3.10. *Any basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of $H_1(M)$ with the following intersection properties*

$$a_j \cdot b_k = \delta_{jk}$$
$$a_j \cdot a_k = 0 = b_j \cdot b_k,$$

for all $j, k = 1, \ldots, g$ is called a canonical homology basis *for M.*

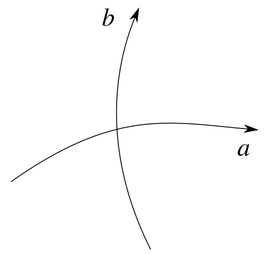


Figure 2: Intersection of curves that contributes by 1 to the intersection index $a \cdot b$

Every curve (its equivalence class to be precise) can be decomposed into the curves of a canonical homology basis by

$$c = \sum_{j=1}^{g} m_j a_j + n_j b_j,$$

where $m_j, n_j \in \mathbb{Z}$ are given by

$$m_j = c \cdot b_j, \quad n_j = a_j \cdot c. \tag{3.5}$$

3.3 Differential forms

Let M be a Riemann surface. A *0-form* on M is a function on M. A *differential form* (*1-form* or *differential*) is an assignment of two continuous functions f and g to each local coordinate $z = x + \mathrm{i}y$ on M such that

$$f\ dx + g\ dy \tag{3.6}$$

is invariant under coordinate changes. A *2-form* on M is an assignment of a continuous function f to each local coordinate z such that

$$f\ dx \wedge dy$$

is invariant under coordinate changes. We shall consider differential forms of the type

$$u(z)dz + v(z)d\bar{z}, \tag{3.7}$$

where

$$dz = dx + \mathrm{i}\ dy, \qquad d\bar{z} = dx - \mathrm{i}\ dy,$$

and comparing with (3.6) we get

$$f = u + v, \qquad g = \mathrm{i}(u - v).$$

A 1-form ω can be integrated over a finite union of paths. Thus, if the piece-wise differentiable path c is contained in a single coordinate disc $z = x + \mathrm{i}y$, $c : I \to M$, $I = [0, 1]$, and if ω is given by (3.6), then

$$\int_c \omega = \int_0^1 \left\{ f(x(t), y(t))\frac{dx}{dt} + g(x(t), y(t))\frac{dy}{dt} \right\} dt.$$

The integral is independent of the choice of z.

For forms with $\mathcal{C}^1$ coefficients, we introduce the differential operator d. For $\mathcal{C}^1$ functions f we define

$$df = f_x\, dx + f_y\, dy = f_z\, dz + f_{\bar{z}}\, d\bar{z},$$

where $f_z = f_x - \mathrm{i}f_y$ and $f_{\bar{z}} = f_x + \mathrm{i}f_y$. For a $\mathcal{C}^1$ form ω given by (3.7) we define

$$d\omega = (v_z - u_{\bar{z}})dz \wedge d\bar{z}.$$

Definition 3.11. *A 1-form ω is called* exact *if $\omega = df$ for some $\mathcal{C}^2$ function f on M. ω is called closed if it is $\mathcal{C}^1$ and $d\omega = 0$. A 1-form ω is called* holomorphic *provided that locally $\omega = df$ with f holomorphic.*

A differential ω of the form (3.7) *is holomorphic if and only if $v = 0$ and u is a holomorphic function of the local coordinate.*

A meromorphic differential *(*abelian differential*) on a Riemann surface is an assignment of a meromorphic function f to each local coordinate z such that*

$$f(z)\, dz \tag{3.8}$$

is invariant under coordinate changes.

For an abelian differential ω we define the *residue* of ω at P by

$$\mathrm{res}_P\, \omega = a_{-1},$$

where ω is given by (3.8) in terms of the local coordinate z that vanishes at P, and the Laurent series of f is

$$f(z) = \sum_{n=N}^{\infty} a_n z^n.$$

The smallest n such that $a_n \neq 0$ is called *order of f at P* and is denoted by $\mathrm{ord}_f\, P$.

The abelian differentials which are holomorphic will be called of the *first kind*, while the meromorphic abelian differentials with zero residues will be called of the *second kind*. A general abelian differential (which may have residues) will be called of the *third kind*. We define *abelian integral* $\Omega(P)$ on a Riemann surface M as a meromorphic function that is given as an integral of an abelian differential $d\Omega$

$$\Omega(P) = \int_{P_0}^{P} d\Omega, \quad P \in M,$$

where $P_0 \in M$ is some point. The abelian integral $\Omega(P)$ will be called of the first, second or third kind according to the abelian differential.

3.4 Abelian differentials on compact surfaces

On a compact Riemann surface M of genus g, the vector space $\mathcal{H}^1(M)$ of holomorphic differentials has the dimension g. For a canonical homology basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$, there exists a unique basis $\{\omega_1, \ldots, \omega_g\}$ for the space of holomorphic abelian differentials (space $\mathcal{H}^1(M)$) with the property

$$\int_{a_j} \omega_k = \delta_{jk}, \qquad j, k = 1, \ldots, g. \tag{3.9}$$

The matrix $\mathit{\Pi} = \pi_{jk}$ with

$$\pi_{jk} = \int_{b_j} \omega_k \tag{3.10}$$

is symmetric with a positive definite imaginary part.

If $\{a', b'\} = \{a_1, \ldots, a'_g, b'_1, \ldots, b'_g\}$ is another canonical homology basis, and $\omega' = \{\omega_1, \ldots, \omega_g\}$ is the basis for $\mathcal{H}^1(M)$ dual to this basis, the new basis can be decomposed into the old basis by

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and the new matrix Π' of b'-periods is given by

$$\Pi' = (C + D\Pi)(A + B\Pi)^{-1}. \tag{3.11}$$

3.5 Divisors

For this section, let M be a compact Riemann surface of genus $g \geq 0$.

Definition 3.12 (Divisor). *A* divisor *on M is a formal symbol*

$$\mathfrak{A} = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_k^{\alpha_k},$$

with $P_j \in M$, $\alpha_j \in \mathbb{Z}$.

We can write the divisor $\mathfrak{A}$ as

$$\mathfrak{A} = \prod_{P \in M} P^{\alpha(P)}, \tag{3.12}$$

where $\alpha(P) \in \mathbb{Z}$, $\alpha(P) \neq 0$ for only finitely many $P \in M$.

We introduce the *group of divisors* on M denoted by $\mathrm{Div}(M)$. It is a commutative group (written multiplicately, i. e. $\mathfrak{A}\mathfrak{B} = \prod_{P \in M} P^{\alpha(P)+\beta(P)}$) on the points in M. The unit element of the group $\mathrm{Div}(M)$ is denoted by 1 (divisor with no points, $\alpha(P) = 0$ for $P \in M$).

For $\mathfrak{A} \in \mathrm{Div}(M)$ given by (3.12), we define

$$\deg \mathfrak{A} = \sum_{P \in M} \alpha(P).$$

If $f \in \mathcal{K}(M) \setminus \{0\}$, then f determines a divisor $(f) \in \mathrm{Div}(M)$ by

$$(f) = \prod_{P \in M} P^{\mathrm{ord}_P f}.$$

A divisor in the image of $(\cdot) : \mathcal{K}(M) \to \mathrm{Div}(M)$ is called *principal.* The group of divisors modulo principal divisors is known as the *divisors class group.* Divisors $\mathfrak{A}$, $\mathfrak{B}$ are called *equivalent* ($\mathfrak{A} \sim \mathfrak{B}$) provided that $\mathfrak{A}\mathfrak{B}^{-1}$ is principal, where $\mathfrak{B}^{-1} = \prod_{P \in M} P^{-\beta(P)}$.

The divisor $\mathfrak{A}$ of (3.12) is *integral* ($\mathfrak{A} \geq 1$) provided that $\alpha(P) \geq 0$ for all P. This introduces a partial ordering on divisors: $\mathfrak{A} \geq \mathfrak{B}$ if and only if $\mathfrak{A}\mathfrak{B}^{-1} \geq 1$.

For every integer $n \geq 1$, M_n stands for the set of the integral divisors of degree n on M.

A function $f \in \mathcal{K}(M)$, $f \neq 0$, is said to be a *multiple* of a divisor $\mathfrak{A}$ provided that $(f) \geq \mathfrak{A}$. For a simplification, we assume convention that $(0) \geq \mathfrak{A}$ for all divisors $\mathfrak{A} \in \mathrm{Div}(M)$.

For a divisor $\mathfrak{A}$ on M, we set

$$L(\mathfrak{A}) = \{f \in \mathcal{K}(M) \mid (f) \geq \mathfrak{A}\},$$

the vector space of functions that are multiples of the divisor $\mathfrak{A}$. Its dimension is denoted $r(\mathfrak{A})$. We also set

$$\Omega(\mathfrak{A}) = \{\tau \mid \tau \text{ is an abelian differential on } M \text{ with } (\tau) \geq \mathfrak{A}\},$$

the vector space of abelian differentials that are multiples of the divisor $\mathfrak{A}$. Its dimension is denoted $i(\mathfrak{A})$. The most important result on divisors is the Riemann-Roch theorem

Theorem 3.3 (Riemann-Roch)**.** *Let M be a compact Riemann surface of genus g and $\mathfrak{A}$ be any divisor on M, Then*

$$r(\mathfrak{A}^{-1}) = \deg \mathfrak{A} - g + 1 + i(\mathfrak{A}).$$

Proof. See [9], III.4. □

Definition 3.13. *Divisor D is called* special *provided that $r(D^{-1}) > 1$.*

3.6 Abelian mapping

Let M be a compact Riemann surface of genus $g > 0$.

We define the lattice $L(M)$ generated by the $2g$-columns of the $g \times 2g$ matrix (I, Π), where I is the $g \times g$ identity matrix. Denote these columns by $e^{(1)}, \ldots, e^{(g)}, \pi^{(1)}, \ldots, \pi^{(g)}$. These vectors are clearly linearly independent over the reals[11]. The lattice $L(M)$ is the set

$$L(M) = \left\{ \sum_{j=1}^{g} m_j e^{(j)} + \sum_{j=1}^{g} n_j \pi^{(j)} \mid m, n \in \mathbb{Z}^g \right\}.$$

The lattice $L(M)$ introduces in $\mathbb{C}^g$ the equivalence relation

$$x \sim y \Leftrightarrow x - y \in L(M).$$

The quotient of $\mathbb{C}^g$ by this equivalence relation, $J(M) = \mathbb{C}^g / L(M)$, is called the *Jacobian variety of M*. We define a map

$$\varphi : M \to J(M)$$

by choosing a point $P_0 \in M$ and setting

$$\varphi_{P_0}(P) = \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g \right).$$

This map is called the *Abelian mapping.* We shall drop the subscript P_0 if it is clear what base point we are using. We can extend the map φ to integral divisors on M ,

$$\varphi : M_n \to J(M)$$

[11] $e^{(k)} \in \mathbb{R}^g$ are vectors of the standard basis of $\mathbb{R}^g$ and since $\Im \Pi > 0$, the vectors $\pi^{(k)}$ are independent over $\mathbb{R}$. Furthermore, $\Im \pi^{(k)} \neq 0$ for all k, thus $e^{(k)}$ and $\pi^{(k)}$ are all linearly independent over the reals.

by setting

$$\varphi(D) = \sum_{j=1}^{n} \varphi(P_j)$$

for $D \in M_n$, $D = P_1 \cdots P_n$.

For general divisor $D \in \mathrm{Div}(M)$ of the form (3.12), we set

$$\varphi(D) = \sum_{P \in M} \alpha(P)\varphi(P).$$

A point $x \in J(M)$ is called *a point of order 2* when $x \neq 0$ and $2x = 0$ in $J(M)$.

3.7 Hyperelliptic surfaces

We will be interested in the simplest Riemann surfaces, so called hyperelliptic Riemann surfaces. In this section we will briefly discuss some of their important properties.

Definition 3.14. *A compact Riemann surface M is called* hyperelliptic *provided that there exists an integral divisor D on M with*

$$\deg D = 2, \quad r(D^{-1}) \geq 2.$$

Equivalently, M is hyperelliptic if and only if M admits a non-constant meromorphic function with precisely 2 poles.

If a Riemann surface M of genus g admits such a function f with exactly two poles, the degree of f is 2 and f is a 2-sheeted cover of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by M. Since the only possible branch numbers of f are 0 or 1 at each point (by virtue of Theorem 3.1), all branch points have the branching number 1. By the Riemann-Hurwitz relation (Theorem 3.2), the total branching number of f is $2g + 2$ and we can therefore describe a hyperelliptic surface of genus g as a two-sheeted covering of the sphere branched at $2g + 2$ points.

Every surface of genus ≤ 2 is hyperelliptic. Surfaces of genus 1 (tori) are often called *elliptic*.

Let λ be a meromorphic function of degree 2 on a hyperelliptic surface M and let $P_1, \ldots, P_{2g+2}$ be the branch points of λ, without the loss of generality such that

$$\lambda(P_j) \neq \infty, \qquad j = 1, \ldots, 2g + 2.$$

The function

$$\mu = \sqrt{\prod_{j=1}^{2g+2} (\lambda - \lambda(P_j))} \tag{3.13}$$

is a meromorphic function on M. This function can be used for construction of holomorphic differentials on M. One can verify that the g differentials

$$\nu_j = \frac{\lambda^j d\lambda}{\mu}, \qquad j = 0, \ldots, g-1, \tag{3.14}$$

form a basis for the Abelian differentials of the first kind on M.

We can also proceed in the other direction. We can construct a hyperelliptic Riemann surface using the two functions λ and μ. In this case, the hyperelliptic surface of genus g is defined as the set of points of $\mathbb{C}^2$ defined by equation

$$\mu^2 = \prod_{j=1}^{2g+2} (\lambda - E_j) \tag{3.15}$$

with

$$E_j \in \mathbb{C}, E_j \neq E_k, \qquad j, k = 1, \ldots, 2g+2, j \neq k.$$

The local parametrization is defined by the homeomorphism $(\mu, \lambda) \mapsto \lambda$ in the neighborhoods of the points (μ_0, λ_0) with $\lambda_0 \neq E_j, \forall j$ and by the homeomorphism $(\mu, \lambda) \mapsto \sqrt{\lambda - E_j}$ in the neighborhood of each point $(0, E_j)$. For this hyperelliptic curve there are two points at infinity and they will be denoted by the symbols ∞^+ and ∞^-. They are distinguished by the conditions

$$(\mu, \lambda) \to \infty^{\pm} \Leftrightarrow \lambda \to \infty, \mu \sim \pm\lambda^{g+1}, \tag{3.16}$$

and the local parameter in the neighborhoods of both points is given by the homeomorphism $(\mu, \lambda) \mapsto \lambda^{-1}$.

To include all general surfaces, we can allow one of the points E_j (without the loss of generality $j = 2g+2$) to be the complex infinity[12]. In this case the equation (3.15) has the form

$$\mu^2 = \prod_{j=1}^{2g+1} (\lambda - E_j)$$

with the same assumptions on E_j except $E_{2g+2} = \infty$ now. In this case there is only one point at infinity denoted by ∞, it is distinguished by the condition

$$(\mu, \lambda) \to \infty \Leftrightarrow \lambda \to \infty, \mu \sim \lambda^{g+\frac{1}{2}}$$

and the local parameter in its neighborhood is given by the homeomorphism $(\mu, \lambda) \mapsto \lambda^{-1/2}$.

[12]Points E_j correspond to $\lambda(P_j)$ in (3.13). Since λ is a meromorphic function of degree 2, one of its branch points can be in general at ∞.

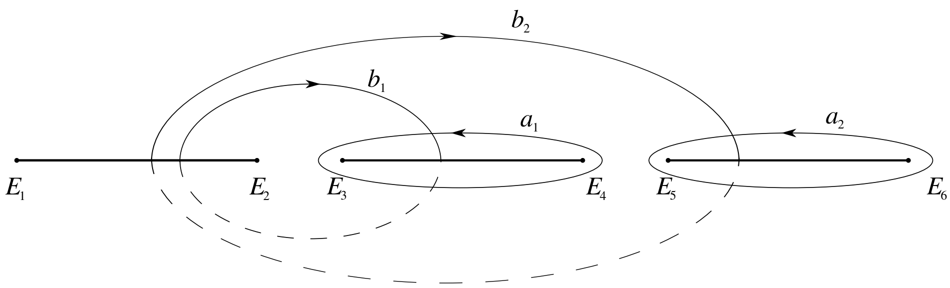


Figure 3: Construction of canonical homology basis for a hyperelliptic surface using the two-sheeted covering of the Riemann sphere

A construction of the canonical homology basis for a hyperelliptic surface M is a simple task when we know the two-sheeted covering of the Riemann sphere by M. In this case, we can construct the basis on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ as seen in figure 3.

4 Theta Functions

The most important part of the theory of Riemann surfaces is the construction of meromorphic functions. The theory of theta functions is a very good tool for it. In this section we will review definitions and some basic properties of theta functions and especially of theta functions associated with a Riemann surface. All definitions are based on the book by Farkas & Kra [9]. More properties of theta functions can be found in Rodin [37].

4.1 Riemann theta function

For fixed integer $g \geq 1$, the symbol $\mathbb{H}_g$ denotes the *Siegel upper half space of genus g*. It is the space of complex symmetric $g \times g$ matrices with positive definite imaginary part

$$\mathbb{H}_g = \left\{ \tau \in \mathbb{C}^{g \times g} \mid \tau = \tau^T, \Im\tau > 0 \right\},$$

where T denotes the matrix transposition. It is an open subset of the manifold of complex symmetric $g \times g$ matrices.

We define *Riemann's theta function* by

$$\theta(z, \tau) = \sum_{N \in \mathbb{Z}^g} \exp 2\pi \mathrm{i} \left(\frac{1}{2} N^T \tau N + N^T z \right), \tag{4.1}$$

where $z \in \mathbb{C}^g$ is a complex vector and $\tau \in \mathbb{H}_g$ is a complex symmetric matrix with positive definite imaginary part. This function converges on compact subsets of $\mathbb{C}^g \times \mathbb{H}_g$ and it is a holomorphic function [36], §7.

The periodicity of the theta function is the most interesting property. In the following text, I will denote the $g \times g$ identity matrix and $e^{(k)}$ will be its k-th column, and $\tau \in \mathbb{H}_g$ and $\tau^{(k)}$ denote the k-th column of τ.

Theorem 4.1. *Let $\mu, \nu \in \mathbb{Z}^g$. Then*

$$\theta(z + I\nu + \tau\mu, \tau) = \exp 2\pi \mathrm{i} \left[-\mu^T z - \frac{1}{2} \mu^T \tau \mu \right] \theta(z, \tau) \tag{4.2}$$

for all $z \in \mathbb{C}^g$ and $\tau \in \mathbb{H}_g$.

In addition to this periodicity formula, $\theta(z, \tau)$ is an even function of z, that is

$$\theta(-z, \tau) = \theta(z, \tau) \qquad \text{for all } z \in \mathbb{C}^g, \text{all } \tau \in \mathbb{H}_g.$$

We can consider in place of $\theta(z, \tau)$ the translated function $\theta(z + w, \tau)$ for some $w \in \mathbb{C}^g$. We have immediately

$$\theta(z + w + e^{(k)}, \tau) = \theta(z + w)$$

and

$$\theta(z+e+\tau^{(k)},\tau)=\exp 2\pi\mathrm{i}\left[-z_k-e_k-\frac{\tau_{kk}}{2}\right]\theta(z+w,\tau).$$

Because the columns of the $g\times 2g$ period matrix (I,τ) are linearly independent over $\mathbb{R}$ by the same reasons as in section 3.6, every $w\in\mathbb{C}^g$ can be expressed as

$$w=I\frac{\varepsilon'}{2}+\tau\frac{\varepsilon}{2}\equiv\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$$

for some $\varepsilon',\varepsilon\in\mathbb{R}^g$. Thus

$$\theta(z+w,\tau)=\theta\left(z+I\frac{\varepsilon'}{2}+\tau\frac{\varepsilon}{2},\tau\right). \tag{4.3}$$

This suggests[13] that we should define a theta function with characteristics, a function on $\mathbb{C}^g\times\mathbb{H}_g$, by

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau)=$$
$$\sum_{N\in\mathbb{Z}^g}\exp 2\pi\mathrm{i}\left\{\frac{1}{2}\left(N+\frac{\varepsilon}{2}\right)^T\tau\left(N+\frac{\varepsilon}{2}\right)+\left(N+\frac{\varepsilon}{2}\right)^T\left(z+\frac{\varepsilon'}{2}\right)\right\}.$$

The relation with the theta function (4.1) is

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau)=\exp 2\pi\mathrm{i}\left[\frac{1}{8}\varepsilon^T\tau\varepsilon+\frac{1}{2}\varepsilon^Tz+\frac{1}{4}\varepsilon^T\varepsilon'\right]\theta\left(z+I\frac{\varepsilon'}{2}+\tau\frac{\varepsilon}{2},\tau\right), \tag{4.4}$$

in particular

$$\theta\begin{bmatrix}0\\ 0\end{bmatrix}(z,\tau)=\theta(z,\tau).$$

For a construction of solutions to PDEs, the most important case is when ε and ε' are integer vectors ($\varepsilon,\varepsilon'\in\mathbb{Z}^g$). In this case, $\theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](z,\tau)$ is called the *first order theta function with integer characteristic* $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$. The characteristic $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$ is called *non-singular* (resp. *singular*), if $\operatorname{grad}_z\theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](z,\tau)\big|_{z=0}\neq 0$ (resp. $=0$).

4.2 Theta functions associated with a Riemann surface

In the previous section we defined the Riemann's theta function as a holomorphic function defined on $\mathbb{C}^g\times\mathbb{H}_g$. In this section we want to associate the theta function with a compact Riemann surface M.

[13]The theta function with characteristic has the same divisor of zeros as the translated function (4.3).

Let M be a surface of genus $g \geq 1$. Let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a canonical basis on M, $a = \{a_1, \ldots, a_g\}^T$, $b = \{a_1, \ldots, b_g\}^T$, and let $\omega^T = \{\omega_1, \ldots, \omega_g\}$ be a basis for holomorphic differentials $\mathcal{H}^1(M)$ dual to the canonical homology basis.

Since the matrix Π of b-periods defined in section 3.4 is a complex symmetric matrix with positive definite imaginary part, i. e. $\Pi \in \mathbb{H}_g$, we can define first order theta functions with characteristics $\theta \left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](z, \Pi)$ using this matrix.

We will study the theta functions associated with a surface M as a function on M. We will consider the function $\theta \left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](\cdot, \Pi) \circ \varphi$ (for the sake of brevity $\theta \circ \varphi$), where φ is the Abelian mapping $\varphi : M \to J(M)$ introduced in section 3.6 and $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$ is an integer characteristic. Function $\theta \circ \varphi$ is not single valued on M because φ is not single valued as a function $M \to \mathbb{C}^g$ and depends on the path of integration. The function $\theta \circ \varphi$ has a very simple multiplicative behavior in the sense that the continuation of $\theta \left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](\cdot, \Pi) \circ \varphi$ along the closed curve a_j and b_j beginning at a point $P \in M$ multiplies it by

$$\exp 2\pi \mathrm{i} \left[\frac{\varepsilon_j}{2}\right]$$

and

$$\exp 2\pi \mathrm{i} \left[-\frac{\varepsilon'_j}{2} - \frac{\pi_{jj}}{2} - \varphi_j(P)\right],$$

respectively.

Even though $\theta \circ \varphi$ has multiple values, the set of zeros is a well defined set on M. There are two possibilities only. Either $\theta \circ \varphi$ vanishes identically on M or it has only a finite number of zeros on M. By evaluating the integral

$$\frac{1}{2\pi \mathrm{i}} \int d \log \theta \circ \varphi$$

along a curve constructed from all the curves in the canonical homology basis (in both directions, so called fundamental polygon of M, for details see VI.2.4 in [9]) and by taking care of the analytic continuation of the integrand, we come to exactly g zeros of the function $\theta \circ \varphi$ (counting multiplicities).

The set of zeros of a theta function is described in a quite detailed way by the Riemann theorem on the divisor of zeros of the theta function. The formulation presented here is from [1], 2.7. More general formulations together with proofs can be found in [9], VI.3.

Theorem 4.2 (Riemann Theorem)**.** *Let M be a compact Riemann surface with a canonical basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ and let $\mathcal{K} \in \mathbb{C}^g$ be a vector of*

Riemann constants

$$\mathcal{K}_j = \frac{\pi_{jj}}{2} - \sum_{k=1}^{g} \int_{a_k} \varphi_j \omega_k$$

for $j = 1, \ldots, g$. *Let* $w \in J(M)$ *be a vector such that the Riemann theta function* $F(P) = \theta(\varphi(P) - w - \mathcal{K}, \Pi)$ *does not vanish identically on* M. *Then*

(i). the function $F(P)$ *has on* M *exactly* g *zeros* $P_1, \ldots, P_g$ *that give a solution to the Jacobi inversion problem*

$$\varphi(P_1 \cdots P_g) = w \tag{4.5}$$

(ii). the divisor $D = P_1 \cdots P_g$ *is nonspecial (see section 3.5),*

(iii). the points $P_1, \ldots, P_g$ *are defined from* (4.5) *uniquely up to a permutation.*

This theorem implies that for the nonspecial divisor $D = P_1 \cdots P_g$, the function $F(P) = \theta(\varphi(P) - w - \mathcal{K}, \Pi)$ has on M exactly g zeros $P = P_1, \ldots, P_g$.

Remark. The definition of the Riemann's theta function (4.1) is different in some publications (for example [1]). The difference is in the convention of definition of the b-period matrix Π. A canonical homology basis is often normalized, in contrast to (3.9), by the conditions

$$\int_{a_j} \tilde{\omega_k} = 2\pi \mathrm{i} \delta_{jk}$$

and this changes the properties of the b-period matrix $\tilde{\Pi}$, namely in this case it is a complex symmetric matrix with negative definite real part. By definition (3.10), the relation of b-period matrices is simply $\tilde{\Pi} = 2\pi\mathrm{i}\Pi$ then. The Riemann's theta function has therefore to be defined as

$$\tilde{\theta}(\tilde{z}, \tilde{\tau}) = \sum_{N \in \mathbb{Z}^g} \exp\left(\frac{1}{2} N^T \tilde{\tau} N + N^T z\right),$$

where $\tilde{z} \in \mathbb{C}^g$ is again a complex vector and $\tilde{\tau} \in \mathbb{C}^{g \times g}$ is a complex symmetric matrix with negative definite real part. The arguments of the theta function defined in this way are related to the argument of our theta function by the relations $\tilde{z} = 2\pi\mathrm{i}z$ and $\tilde{\tau} = 2\pi\mathrm{i}\tau$.

5 Riemann-Hilbert problem

The Riemann-Hilbert problem or the Riemann boundary problem is the problem to determine analytic matrices or functions on a Riemann surface that have a certain jump across a contour on the surface. This problem appears in different areas of mathematics and physics. It was formulated by B. Riemann and was studied by D. Hilbert, C. Hazeman, J. Plemelj and others. The Riemann-Hilbert problem has a wide range of physical applications, such as in contact problems of elasticity, dispersion relations in quantum mechanics, flow problems in hydrodynamics, diffraction theory, and so on [36].

Although solution of the general Riemann-Hilbert problem is not known, there are two cases when explicit solution can be given in terms of theta functions on Riemann surfaces and that are useful for generating solutions to the Ernst equation – the scalar Riemann-Hilbert problem (section 5.1) and the Riemann-Hilbert problem with quasi-permutation monodromy matrices [25] (section 5.2).

5.1 Scalar Riemann-Hilbert problem

The solution of this problem was given by S. I. Zverovich (1971) [36]. We present here the version of the problem on a hyperelliptic surface $\mathcal{L}_{\mathrm{H}}$.

Let $\mathcal{L}_{\mathrm{H}}$ be a hyperelliptic Riemann surface of genus g with branch points with projections $\xi, \bar{\xi}$, $E_1, F_1, \ldots, E_g, F_g$ on $\mathbb{CP}^1$. Let $\theta(z)$ be the theta function on $\mathcal{L}_{\mathrm{H}}$. Let φ be the Abelian mapping $\mathcal{L}_{\mathrm{H}} \to J(\mathcal{L}_{\mathrm{H}})$ with a base point p_0 with the projection ξ on $\mathbb{CP}^1$. Let p denote a point on $\mathcal{L}_{\mathrm{H}}$.

Let Γ be a piecewise smooth contour without self-intersections on $\mathcal{L}_{\mathrm{H}}$. Let $\Lambda = p_1 \cdots p_r$ be a divisor on Γ consisting of a finite number of mutually different points subject to the condition that $\Gamma \setminus \Lambda$ decomposes into a finite set of connected components $\{\Gamma_j\}$, $j = 1, \ldots, N$, each of which is homeomorphic to the interval $(0, 1)$. Each Γ_j has a starting and an end point, given by two points of Γ. We define the functions $\alpha(p, \Gamma_j)$ for $p \in \Gamma$ by

$$\alpha(p, \Gamma_j) = \begin{cases} 1 & \text{if } p \in \Gamma_j \\ 0 & \text{otherwise} \end{cases},$$

for $j = 1, \ldots, N$. On each curve Γ_j let there be defined a Hölder continuous function $G_j(p)$, which is finite and nonzero. We will denote

$$G(p) = \sum_{j=1}^{N} \alpha(p, \Gamma_j) G_j(p), \quad p \in \Gamma \setminus \Lambda. \tag{5.1}$$

Let there be a divisor $\mathcal{A}$ of degree m, consisting of points of the divisor Λ, taken at arbitrary degree. Let on $\mathcal{L}_{\rm H}\setminus\Gamma$ be another divisor $\mathcal{B}$ of degree n.

Definition 5.1. *The* homogeneous scalar Riemann-Hilbert problem *is formulated as follows: Determine a function ψ in $\mathcal{L}_{\rm H}\setminus\Gamma$ holomorphic and continuous up to the boundary satisfying the boundary condition on Γ,*

$$\psi^+(p) = G(p)\psi^-(p), \quad p\in\Gamma, \tag{5.2}$$

with $\psi \in L(\mathcal{A}^{-1}\mathcal{B}^{-1})$.

The key point in the construction of the solution to the problem 5.1 is an analogue to the usual Cauchy kernel $\frac{1}{t-z}$. This analogue is given by a normalized Abelian differential of the third kind with poles at p and p_0 and is denoted by ω_{pp_0} (see sect. 3.4). The solution to the problem 5.1 is then given by [21, 36]

$$\tilde{\psi}(p) = e^{\gamma(p)}, \tag{5.3}$$

with

$$\gamma(p) = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \ln G\ \omega_{pp_0}, \tag{5.4}$$

where $p_0\notin\Gamma$. The function $\tilde{\psi}(p)$, however, is not single-valued on $\mathcal{L}_{\rm H}$ but it has multiplicative periods along the cycles b_j given by [36], p. 39,

$$\tilde{\psi}_j = \exp\left(\int_\Gamma \ln G\ \omega_j\right). \tag{5.5}$$

The function $\tilde{\psi}(p)$ changes its value when continued along the cycle b_j as $\tilde{\psi}(p)\to\tilde{\psi}_j\tilde{\psi}(p)$. One can check that (5.3) solves problem 5.1 by applying the Plemelj-Sokhotsky formulae [36], p. 26,

$$\gamma(p)^{\pm} = \pm\frac{1}{2}\ln G(p) + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \ln G\ \omega_{pp_0}, \quad p\in\Gamma\setminus\Lambda, \tag{5.6}$$

where $\gamma^{\pm}(p)$ denotes the limiting values of $\gamma(p)$ at Γ. Therefore

$$\tilde{\psi}^+(p) = \exp\left(\frac{1}{2}\ln G(p) + \frac{1}{2\pi \mathrm{i}}\int_\Gamma \ln G\ \omega_{pp_0}\right) = \exp(\ln G(p))\tilde{\psi}^-(p),$$

for $p\in\Gamma\setminus\Lambda$.

Since the function $\tilde{\psi}(p)$ is not single-valued on $\mathcal{L}_{\rm H}$, we need to modify it to be useful as a solution of the Ernst equation. Let $p_0\in\mathcal{L}_{\rm H}$ be a fixed point not coinciding with the singularities of $\tilde{\psi}$ or the branch points E_i and

F_i. Let $\Omega(p)$ be a linear combination of normalized[14] Abelian integrals of the second kind and of the third kind such that the singularities of Ω do not coincide with the branch points E_i and F_i and are independent of ξ and $\bar{\xi}$. Let $D = p_1 + \cdots + p_g$ be a fixed non-special divisor on $\mathcal{L}_{\mathrm{H}}$. Let $2\pi\mathrm{i}v$ be the vector of b-periods of the Abelian differential $d\Omega$ given by

$$v_j = \int_{b_j} d\Omega(p).$$

Let $2\pi\mathrm{i}u$ be the vector of additive b-periods of $\gamma(p)$ (5.4) given using (5.5) as

$$u_j = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \ln G \; \omega_j.$$

Using all these assumptions, the following theorem was formulated and proved in [21], for example.

Theorem 5.1. *The generalized solution to the scalar Riemann-Hilbert problem 5.1 with the assumptions above is given by the function*

$$\psi(p) = \psi_0 \frac{\theta(\varphi(p) - \varphi(D) + u + v - \mathcal{K})}{\theta(\varphi(p) - \varphi(D) - \mathcal{K})} \times \exp\left\{ \Omega(p) + \frac{1}{2\pi\mathrm{i}} \int_\Gamma \ln G \; \omega_{pp_0} \right\}, \quad (5.7)$$

where ψ_0 is a normalization constant and $\mathcal{K}$ is the vector of Riemann constants from theorem 4.2. The paths of integration have to be the same for all integrals in φ and Ω.

Proof. Since the function $\tilde{\psi}(p)$ is already a solution to the scalar Riemann-Hilbert problem 5.1, it remains to prove that $\psi(p)$ is a single-valued function on $\mathcal{L}_{\mathrm{H}}$. If we continue $\psi(p)$ along a closed loop c that can be decomposed in the canonical homology basis of $\pi_1(\mathcal{L}_{\mathrm{H}})$ as

$$c = \sum_{j=1}^{g} m_j a_j + \sum_{j=1}^{g} n_j b_j = M^T a + N^T b,$$

with $m_j, n_j \in \mathbb{Z}$ and $M = (m_1, \ldots, m_g)$, $N = (n_1, \ldots, n_g)$, the integrals change values according to the formulas

$$\begin{aligned} \varphi(p) &\to \varphi(p) + M + \Pi N, \\ \Omega(p) &\to \Omega(p) + 2\pi\mathrm{i} N^T v, \\ \gamma(p) &\to \gamma(p) + 2\pi\mathrm{i} N^T u. \end{aligned}$$

[14] $\int_{a_j} d\Omega(p) = 0$.

Under this transformation, the theta function transforms according to the periodicity property (4.2) and we observe that $\psi(p)$ transforms as

$$\psi(p) \to \psi(p) \exp(-2\pi \mathrm{i} N^T(u+v)) \exp(2\pi \mathrm{i} N^T(u+v)) = \psi(p).$$

Thus the function (5.7) is a single valued function on $\mathcal{L}_{\mathrm{H}}$. □

The function $\psi(p)$ has g simple poles on $\mathcal{L}_{\mathrm{H}}$ at the points $p_1, \ldots, p_g$ from the Riemann theorem 4.2 and g simple zeros. Additional poles, zeros, and essential singularities can be obtained by a suitable choice of Abelian integrals in Ω. Without loss of generality, we can choose D to consist only of branch points of $\mathcal{L}_{\mathrm{H}}$ since D gives the poles of ψ due to the zeros of the theta function in the denominator and this can always be compensated for by a suitable choice of zeros and poles of ψ which arise from the integrals in Ω. All $P \in D$ shall have multiplicity 1.

5.2 Matrix Riemann-Hilbert problem with quasi permutation monodromy matrices

An explicit solution to a general Riemann-Hilbert problem is not known. A quite general class of Riemann-Hilbert problems was, however, solved in a paper by Korotkin in 2004 [25]. It is a Riemann-Hilbert problem to find a square $N \times N$ matrix Ψ on $\mathbb{CP}^1$that has singularities at some points γ_j, and has jumps across some contours connecting γ_j to some fixed point γ_0 that are given by

$$\Psi_+(\gamma) = \Psi_-(\gamma) M(\gamma).$$

Matrices M are assumed to be quasi-permutation $N \times N$ matrices, i. e. matrices with exactly one nonzero element in each column and row. A solution to this Riemann-Hilbert problem can be given on a Riemann surface that is a N-sheeted cover of $\mathbb{CP}^1$ with branch points projected on the points γ_j using a generalization of the Cauchy kernel, the so-called Szegö kernel. This topic will not be pursued further here, for details of the construction of a solution, see [25].

6 Solution of evolution nonlinear systems

In the theory of completely integrable partial differential equations, various types of generating techniques were successfully used to find general classes of solutions that lead to a better understanding of those systems and even to understanding of important physical phenomena such as solitons. Especially algebro-geometric methods are a systematic approach to systems with a known linear system, so called Lax represantation.

6.1 Solutions of the nonlinear Schrödinger equation

The nonlinear Schrödinger (NS) system is one of important systems in theoretical physics. It defines the evolution of complex valued functions $y(x,t)$ and $y^*(x,t)$ by two equations

$$\mathrm{i}y_t + y_{xx} - 2y^*y^2 = 0, \quad (6.1)$$

$$-\mathrm{i}y_t^* + y_{xx} - 2yy^{*2} = 0. \quad (6.2)$$

The Lax representation

$$U_t(\lambda) - V_x(\lambda) = [V(\lambda), U(\lambda)], \qquad \lambda \in \mathbb{C}, \quad (6.3)$$

for this system was first found by Zakharov and Shabat (see [1], p. 87). In this case U and V are 2×2 matrices of the form

$$U(\lambda) \equiv -\mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & \mathrm{i}y \\ -\mathrm{i}y^* & 0 \end{pmatrix}, \quad (6.4)$$

$$V(\lambda) \equiv 2\lambda U(\lambda) + \begin{pmatrix} -\mathrm{i}yy^* & y_x \\ -y_x^* & \mathrm{i}yy^* \end{pmatrix}, \quad (6.5)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli's matrix. The Lax representation (6.3) is equivalent to the compatibility condition of the associated linear system

$$\Psi_x = U\Psi, \quad (6.6)$$

$$\Psi_t = V\Psi \quad (6.7)$$

for the 2×2 matrix function $\Psi = \Psi(\lambda, x, t)$. This function plays an important role in the construction of solutions of the NS system. The standard procedure is an investigation of the most general properties of Ψ that satisfies (6.6) with matrices U and V having the form (6.4). The following theorem summarizes the results from [1]:

Theorem 6.1. *Let $\Psi(\lambda, x, t)$ be a 2×2 matrix function holomorphic in λ in some neighborhood of infinity without the infinity point on the Riemann sphere $\mathbb{CP}^1$ and depending smoothly on x and t, with the asymptotic expansion*

$$\Psi(\lambda, x, t) = \left[I + \sum_{k_1}^{\infty} \Psi_k(x,t)\lambda^{-k} \right] \exp(-\mathrm{i}\lambda x\sigma_3 - 2\mathrm{i}\lambda^2 t\sigma_3)C(\lambda)$$

at $\lambda = \infty$ that can be differentiated by terms with respect to x and t, where $C(\lambda) \in GL(2, \mathbb{C})$ is constant in x and t. Then the logarithmic derivatives of Ψ have the asymptotic behavior

$$\begin{aligned} \Psi_x\Psi^{-1} &= U(\lambda) + O(\lambda^{-1}), \\ \Psi_t\Psi^{-1} &= V(\lambda) + O(\lambda^{-1}) \end{aligned}$$

as $\lambda \to \infty$ and the matrices U and V are of the form (6.4). *Functions y and y^* that are the solution of* (6.1) *are proportional to non-diagonal elements of the matrix $\Psi_1(x,t)$,*

$$\begin{aligned} y(x,t) &= 2(\Psi_1)_{12}, \\ y^*(x,t) &= 2(\Psi_1)_{21}. \end{aligned}$$

To construct a solution Ψ of the linear system, we fix an arbitrary hyperelliptic Riemann surface X given by the relation

$$\mu^2 = \prod_{j=1}^{2g+2} (\lambda - E_j),$$

where $E_j \in \mathbb{C}$ are branch points of X, $E_j \neq E_k$ whenever $j \neq k$. We also need the Abelian integrals $\Omega_1(P)$, $\Omega_2(P)$, $\Omega_3(P)$, $P \in X$ which are fixed by the conditions

$$\begin{aligned} \int_{a_j} d\Omega_k &= 0, \quad \forall j,k, \\ \Omega_1(P) &= \pm(\lambda + O(1)), \\ \Omega_2(P) &= \pm(2\lambda^2 + O(1)), \\ \Omega_3(P) &= \pm(\ln\lambda + O(1)) \end{aligned}$$

as $P \to \infty^{\pm}, \lambda = \lambda(P)$ is the projection of P on $\mathbb{CP}^1$, and $\Omega_j(P)$ have no singularities at the points different from $\infty^{\pm}$. Let D be an arbitrary divisor with $\deg D = g$ of general position, i. e. $D = \prod_{i=j}^{g} P_j$, $\lambda(P_j) \neq E_k$, $\forall j,k$, and $j \neq k \Rightarrow \lambda(P_j) \neq \lambda(P_k)$.

The function Ψ can be given in terms of the vector-valued Baker-Akhiezer function $\psi(P,x,t) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ that is uniquely determined by two conditions:

1. $\psi(P)$ is meromorphic on $X \setminus \{\infty^\pm\}$ and its divisor of poles coincides with D.
2. condition for the asymptotic behavior of ψ at $\infty^\pm$ with essential singularities at $\infty^\pm$

$$\begin{aligned}\psi(P) &= \left[\begin{pmatrix}1\\0\end{pmatrix} + O(\lambda^{-1})\right]\exp(-\mathrm{i}\lambda x - 2\mathrm{i}\lambda^2 t), \quad P \to \infty^- \\ \psi(P) &= \alpha\lambda\left[\begin{pmatrix}0\\1\end{pmatrix} + O(\lambda^{-1})\right]\exp(\mathrm{i}\lambda x + 2\mathrm{i}\lambda^2 t), \quad P \to \infty^+, \alpha \in \mathbb{C},\end{aligned}$$

where λ is the projection $\lambda = \lambda(P)$.

The Baker-Akhiezer function is given by the formula

$$\psi_0(P) = c\frac{\theta(\varphi(P)) + v - \varphi(D) - \mathcal{K})}{\theta(\varphi(P)) - \varphi(D) - \mathcal{K})}\exp\Omega(P),$$

where φ is the Abelian mapping on X with the base point $P_0 \in X$, D is a divisor of degree g, $\Omega(P)$ is an Abelian integral of the second kind with poles at some points $Q_1, \ldots, Q_n$, the principal parts of $\Omega(P)$ coincide with some polynomials $q_j(z_j)$, $j = 1, \ldots, n$, where z_j is a local coordinate at Q_j, and v a vector of b-periods of the integrals $\Omega(P)$,

$$v_j = \int_{b_j} d\Omega(P), \quad j = 1, \ldots, g.$$

The path of integration in $\Omega(P)$ and $\varphi(P)$ is chosen to be the same. The Baker-Akhiezer function has poles only at points of the divisor D and in neighborhood of every pont Q_j the estimate

$$\psi_0 \exp(-q_j(z_j(P))) = O(1)$$

holds. Theorem 2.24 in [1] summarizes the properties of Baker-Akhiezer functions. It also states that a Baker-Akhiezer function with some divisor D and polynomials q_j is unique up to a constant $c \in \mathbb{C}$.

A Baker-Akhiezer function $\psi(P, x, t)$ used to solve the nonlinear Schrödinger system is constructed using the Abelian integrals $\Omega_1(P)$, $\Omega_2(P)$, $\Omega_3(P)$ and the asymptotic behavior of ψ. The matrix $\Psi(P)$ is then given by

$$\Psi(\lambda) = (\psi(P^+), \psi(P^-)),$$

where $P^\pm \in X$ are the two preimages of λ so that $P^\pm \to \infty^\pm$ when $\lambda \to \infty$. The resulting formulas are quite long and the topic is not pursued further here. Details of the construction are given in [1], chapt. 4. The construction procedure is, however, very similar to that we will use in section 7.

7 Solutions of the Ernst equation

First, let us show that the solutions of the Ernst equation can be given as solutions to certain Riemann-Hilbert problems on hyperelliptic Riemann surfaces. We will review this approach here.

7.1 Linear system for the Ernst equation

The Ernst equation is completely integrable (see, for example, Maison [29]). This means that it can be considered as the integrability condition of an overdetermined linear differential system for a matrix-valued function Φ that contains an additional variable, the so-called spectral parameter. Several forms of the linear system are known in the literature which are related through gauge transformations (e. g. [4]). The choice of a specific form of the linear system is equivalent to a gauge fixing.

In the literature, the most common form of the associated linear problem is the system for 2×2 matrix function $\Psi(K,\xi,\bar\xi)$,

$$\Psi_\xi\Psi^{-1} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} + \frac{K-\bar\xi}{\mu_0}\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \equiv V, \tag{7.1}$$

$$\Psi_{\bar\xi}\Psi^{-1} = \begin{pmatrix} \overline{N} & 0 \\ 0 & \overline{M} \end{pmatrix} + \frac{K-\xi}{\mu_0}\begin{pmatrix} 0 & \overline{N} \\ \overline{M} & 0 \end{pmatrix} \equiv W. \tag{7.2}$$

The functions M and N depend only on ξ and $\bar\xi$, but not on K, and have the form

$$M = \frac{\overline{\mathcal{E}}_\xi}{\mathcal{E}+\overline{\mathcal{E}}}, \quad N = \frac{\mathcal{E}_\xi}{\mathcal{E}+\overline{\mathcal{E}}},$$

where $\mathcal{E}$ is the Ernst potential. In [34] it is shown that the compatibility condition $\Psi_{\xi\bar\xi} = \Psi_{\bar\xi\xi}$ of the linear system is the Ernst equation. The spectral parameter K lives on the Riemann surface $\mathcal{L}$ of genus 0 given by the relation

$$\mu_0^2(K) = (K-\xi)(K-\bar\xi). \tag{7.3}$$

Points $P \in \mathcal{L}$ are given in the form

$$P \equiv (K, \mu_0(K)), \quad K \in \mathbb{C}.$$

The sign[15] of $\mu_0(K)$ distinguishes the sheet of the cover of $\mathbb{CP}^1$ by $\mathcal{L}$. This is the first indication of the relevance of Riemann surfaces in the context of the Ernst equation. The branch points ξ, $\bar\xi$ of the surface $\mathcal{L}$ depend on the

[15] The branch of the square root $\sqrt{\mu_0^2(K)}$.

spacetime coordinates. This is the special feature of the Ernst equation not present in the theory of evolution equations.

The surface admits two important automorphisms. The holomorphic involution[16] σ interchanging sheets acting on $P \in \mathcal{L}$ by

$$P \equiv (K, \mu_0(K)) \to \sigma(P) \equiv P^\sigma \equiv (K, -\mu_0(K))$$

and the antiholomorphic involution

$$P \equiv (K, \mu_0(K)) \to \tau(P) \equiv \overline{P} \equiv (\overline{K}, \mu_0(\overline{K})).$$

We denote the points $P_0 \equiv (\xi, 0)$ and $\overline{P_0} \equiv (\bar{\xi}, 0)$ on $\mathcal{L}$ simply as ξ and $\bar{\xi}$, respectively. ∞^+ and ∞^- are the points with projection ∞ on $\mathbb{CP}^1$, distinguished by (3.16). In the following, Ψ, $\Psi_\xi \Psi^{-1}$ and $\Psi_{\bar{\xi}} \Psi^{-1}$ stand for the respective functions on $\mathcal{L}$, i. e. $\Psi(P)$, $\Psi_\xi(P)\Psi^{-1}(P)$ ($\equiv \Psi_\xi \Psi^{-1}(P)$), etc. The point $P \in \mathcal{L}$ denotes $(K, \mu_0(K))$. The following theorem is the basic tool in the construction of solutions to (7.1) [20–22].

Theorem 7.1 (Analytic properties of $\Psi(P)$). *Let $\Psi(P)$ be a 2×2 matrix function on $\mathcal{L}$ with the following properties:*

(i). $\Psi(P)$ is holomorphic and invertible at the branch points ξ, $\bar{\xi}$ such that the logarithmic derivative $\Psi_\xi \Psi^{-1}$ diverges[17] *as $1/\sqrt{K-\xi}$ at ξ and $\Psi_{\bar{\xi}} \Psi^{-1}$ diverges as $1/\sqrt{K-\bar{\xi}}$ at $\bar{\xi}$.*

(ii). All singularities of $\Psi(P)$ on $\mathcal{L}$ (poles, essential singularities, zeros of $\det \Psi$, branch-cuts and branch points) are regular, which means that the logarithmic derivatives $\Psi_\xi \Psi^{-1}$ and $\Psi_{\bar{\xi}} \Psi^{-1}$ are holomorphic in the neighborhood of the singular point (thus they have to be independent of ξ and $\bar{\xi}$). In particular, $\Psi(P)$ should have

(a) regular singularities at the points $A_i \in \mathcal{L}$, $i = 1, \ldots, n$, which do not depend on ξ, $\bar{\xi}$,

(b) regular essential singularities at the points $S_i \in \mathcal{L}$, $i = 1, \ldots, m$, which do not depend on ξ,

(c) boundary values at a set of orientable, piecewise smooth contours $\Gamma_i \subset \mathcal{L}$, $i = 1, \ldots, l$, independent of ξ, $\bar{\xi}$ which are related on both sides of the contours via

$$\Psi_-(P) = \Psi_+(P)\mathcal{G}_i(P)|_{P \in \Gamma_i}, \tag{7.4}$$

where $\mathcal{G}_i(P)$ are invertible Hölder continuous matrices independent of ξ, $\bar{\xi}$.

[16] $\sigma^2 = Id$.

[17] $\Psi_\xi \Psi^{-1}$ and $\Psi_{\bar{\xi}} \Psi^{-1}$ have poles of order 1 at ξ, $\bar{\xi}$, respectively.

(iii). $\Psi(P)$ *satisfies the reduction condition*

$$\Psi(P^\sigma) = \sigma_3 \Psi(P)\gamma(P), \tag{7.5}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *is the third Pauli matrix and* $\gamma(P)$ *is a* 2×2 *invertible matrix independent of* ξ, $\bar{\xi}$.

(iv). $\Psi(P)$ *satisfies the normalization and reality condition*

$$\Psi(\infty^+) = \begin{pmatrix} \bar{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix}. \tag{7.6}$$

Then the function $\mathcal{E}$ *in* (7.6) *is a solution of the Ernst equation* (2.4).

Let us remark that the independence of the singular points on ξ, $\bar{\xi}$ means that K is constant in $P = (K, \mu_0(K))$ although $\mu_0(K)$ itself depends on ξ, $\bar{\xi}$.

The theorem was proved in [21] or [18]. Here we present a different proof which is more detailed and precise. In particular, the first part of the proofs in these papers do not give clear arguments why $\Psi_\xi \Psi^{-1}$ and $\Psi_{\bar{\xi}} \Psi^{-1}$ have to be in the form of (7.1). We exploit the Riemann-Roch theorem to prove it here.

Proof. By virtue of (i) and (ii), the function $\Psi_\xi \Psi^{-1}$ is holomorphic on $\mathcal{L} \setminus \{\xi\}$ and its local behavior at ξ is given by

$$\Psi_\xi \Psi^{-1}(P) = \frac{\alpha_{-1}}{\sqrt{K-\xi}} + \alpha_0 + O\left(\sqrt{K-\xi}\right).$$

Since ξ is the branch point of the Riemann surface $\mathcal{L}$, the local parameter at ξ is given by $\sqrt{K-\xi}$ (sect. 3.7). Therefore, $\Psi_\xi \Psi^{-1}(P)$ is a meromorphic function on $\mathcal{L}$ with a pole of order 1 at ξ.

Let D denote the divisor of the point ξ. Because the genus g of $\mathcal{L}$ is 0 and $i(D) \leq i(1) = 0$, the Riemann-Roch theorem 3.3 implies that $r(D^{-1}) = 2$. In other words, all meromorphic functions on $\mathcal{L}$ with a pole of order 1 at ξ are given by a linear combination of a constant function and one particular meromorphic function with a pole of order 1 at ξ. One such function is

$$h(P) = \sqrt{\frac{K-\bar{\xi}}{K-\xi}}, \quad P \equiv (K, \mu_0(K)).$$

Indeed, the function $h(P)$ is a holomorphic function $\mathcal{L} \to \mathbb{CP}^1$. In local coordinates at ξ and $\bar{\xi}$ it is clear that it has a pole of order 1 at ξ and a zero of order 1 at $\bar{\xi}$. Function $\Psi_\xi\Psi^{-1}(P)$ must be thus given by

$$\Psi_\xi\Psi^{-1}(P) = \beta_0\sqrt{\frac{K-\bar{\xi}}{K-\xi}} + \beta_1,$$

where $\beta_0, \beta_1 \in \mathbb{C}^{2\times 2}$ are constant matrices (independent of P, but depending on ξ, $\bar{\xi}$).

The structure of matrices β_0 and β_1 follows from (iii). The involution σ acts on $\Psi_\xi\Psi^{-1}$ as

$$\Psi_\xi(P^\sigma)\Psi^{-1}(P^\sigma) = \sigma_3\Psi_\xi(P)\Psi^{-1}(P)\sigma_3 \tag{7.7}$$

and on $h(P)$ as $h(P^\sigma) = -h(P)$. Only those components of matrix β_0 are nonzero that correspond to the components of $\Psi_\xi\Psi^{-1}$ that change sign under the involution σ in (7.7). Similarly, β_1 has nonzero components where $\Psi_\xi\Psi^{-1}$ has components that do not change sign under σ. Hence, β_0 and β_1 are of the form

$$\beta_0 = \begin{pmatrix} 0 & \times \\ \times & 0 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}.$$

At $P = \infty^+$, the value of $\Psi_\xi\Psi^{-1}$ follows from (iv),

$$\Psi_\xi(\infty^+)\Psi^{-1}(\infty^+) = \frac{1}{\mathcal{E}+\bar{\mathcal{E}}}\begin{pmatrix} \bar{\mathcal{E}}_\xi & \bar{\mathcal{E}}_\xi \\ \mathcal{E}_\xi & \mathcal{E}_\xi \end{pmatrix},$$

and $h(\infty^+) = 1$. Thus

$$\beta_0 + \beta_1 = \frac{1}{\mathcal{E}+\bar{\mathcal{E}}}\begin{pmatrix} \bar{\mathcal{E}}_\xi & \bar{\mathcal{E}}_\xi \\ \mathcal{E}_\xi & \mathcal{E}_\xi \end{pmatrix}$$

and we can conclude that $\Psi_\xi\Psi^{-1}$ has the structure given by the first equation of the linear system (7.1). The structure of $\Psi_{\bar{\xi}}\Psi^{-1}$ is obtained in the same way. □

The solution to the linear system (7.1) is not unique for a given Ernst potential $\mathcal{E}$. It is easy to show that if $\Psi(P)$ is a 2×2 matrix function subject to the conditions of theorem 7.1 and $C(K)$ is a 2×2 matrix function that only depends on K, with properties

$$C(K) = \alpha_1(K)I + \alpha_2(K)\sigma_1,$$
$$\alpha_1(\infty) = 1, \qquad \alpha_2(\infty) = 0,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the first Pauli matrix, then the matrix $\Psi'(P) = \Psi(P)C(K)$ also satisfies the conditions of theorem 7.1 and $\Psi'(\infty^+) = \Psi(\infty^+)$. This matrix leads to the same Ernst potential $\mathcal{E}$ as $\Psi(P)$ [18].

7.2 Construction of the Ernst potential

Theorem 7.1 can be used to construct solutions to the Ernst equation by constructing the matrix function $\Psi(P)$ in accordance with the conditions (i) – (iv). The relevance of the hyperelliptic Riemann surface for construction of $\Psi(P)$ is motivated by a monodromy matrix L, i. e. a matrix function $L(K, \xi, \bar{\xi})$ such that $L\Psi$ is a solution of (7.1) as well. It is a solution to the system

$$L_{\xi} = [W, L], \quad L_{\bar{\xi}} = [V, L].$$

It can be chosen to be traceless and then it is a 2×2 holomorphic matrix function with the structure [21]

$$L = \begin{pmatrix} A(K) & B(K) \\ C(K) & -A(K) \end{pmatrix}$$

and its eigenvalues $\hat{\mu}$ are given as the solution to the equation

$$\hat{\mu}^2(K) = A(K)^2 + B(K)C(K),$$

which is an equation of a Riemann surface. In general, it has infinite genus but we can restrict our analysis to the case of a regular curve with finite genus. Then the Riemann surface is given by an equation of the form

$$\hat{\mu}^2(K) = \prod_{j=1}^{g} (K - E_j)(K - F_j), \tag{7.8}$$

where $E_j, F_j \in \mathbb{C}$ are independent of the physical coordinates $\xi, \bar{\xi}$. This equation represents a two-sheeted covering of the Riemann surface $\mathcal{L}$ and thus a four-sheeted covering of the Riemann sphere $\mathbb{CP}^1$. A point $\hat{P} \in \hat{\mathcal{L}}$ can be given by $\hat{P} \equiv (K, \mu_0(K), \hat{\mu}(K))$. The structure of the surface is showed at the top in figure 4.

It is useful to compute the genus $\hat{g}$ of the surface $\hat{\mathcal{L}}$ using the Riemann-Hurwitz relation, theorem 3.2. $\hat{\mathcal{L}}$ is a 2-sheeted[18] cover of $\mathcal{L}$ which is a surface of genus $\gamma = 0$. Thus the projection $\pi : \hat{\mathcal{L}} \to \mathcal{L}$ is a holomorphic mapping of degree $n = 2$. π is branched at points where $\hat{\mu}(K) = 0$, i. e. at the points $(E_j, \pm\mu_0(E_j), 0)$ and $(F_j, \pm\mu_0(F_j), 0)$ in (7.8). The branch number b_π at these points is 1. Hence the total branching number B is given by the number of these points, $B = 4g$. The Riemann-Hurwitz relation then yields

$$\hat{g} = n(\gamma - 1) + 1 + B/2 = 2g - 1.$$

[18] Another way to compute the genus $\hat{g}$ is to think of $\hat{\mathcal{L}}$ as of the 4-sheeted cover of the Riemann sphere branched at $4g + 4$ points, but it yields the same result for $\hat{g}$.

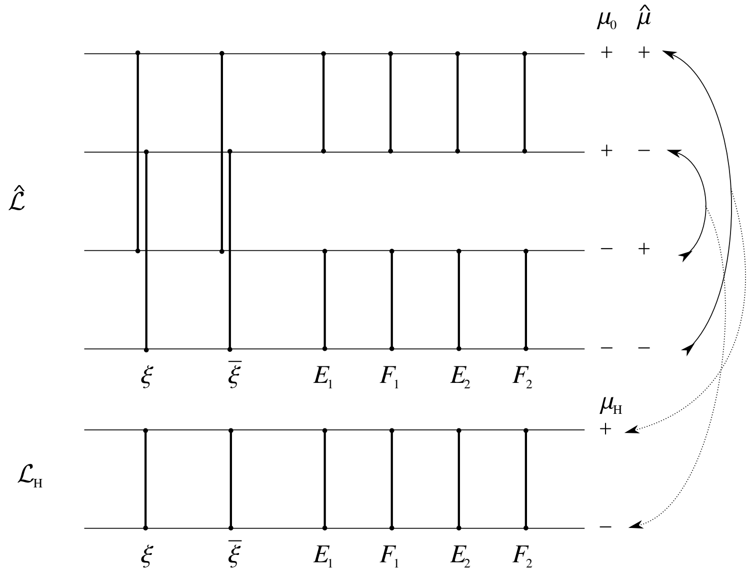


Figure 4: Structure of the Riemann surface $\hat{\mathcal{L}}$ and its quotient $\mathcal{L}_{\mathrm{H}}$

However, the surface $\hat{\mathcal{L}}$ can be factored using an involution on it into a two-sheeted cover of a sphere – a hyperelliptic surface $\mathcal{L}_{\mathrm{H}}$. Then a powerful calculus of hyperelliptic surfaces can be used to study the solution Ψ. This factorization is not as straightforward as is written in [20, 21][19]. The involution we have to use acts on a point of $\hat{\mathcal{L}}$ as

$$\hat{P} \equiv (K, \mu_0(K), \hat{\mu}(K)) \to \hat{\sigma}(\hat{P}) = (K, -\mu_0(K), -\hat{\mu}(K)).$$

Let $H = \{Id, \hat{\sigma}\}$ be a subgroup of the group of automorphisms of $\hat{\mathcal{L}}$. We define the orbit space $\hat{\mathcal{L}}/\hat{\sigma}$ (the quotient of the action of the group H) as the space of the orbits of points $\hat{P} \in \hat{\mathcal{L}}$,

$$H(\hat{P}) = \{h(\hat{P}) \mid h \in H\}.$$

We can topologize the orbit space $\hat{\mathcal{L}}/\hat{\sigma}$ such that the natural projection

$$\pi_{\mathrm{H}} : \hat{\mathcal{L}} \to \hat{\mathcal{L}}/\hat{\sigma}, \quad \pi_{\mathrm{H}}(\hat{P}) \mapsto H(\hat{P}) \tag{7.9}$$

[19] In [20] it is said that the fixed points of the involution become branch points on the factorized surface. This is not correct because the local coordinate at the image of a branch point is z^k, where z is a local coordinate at the branch point and $k = 2$ is the order of the stabilizer subgroup of H [9], III.7.8. That is, the fixed points of the involution remove the branch points of the surface. Another way how to get an insight is to use the Riemann-Hurwitz relation, thorem 3.2, which shows that factorization using an involution with fixed points remove branch points of the surface.

is continuous. Than we introduce a complex structure on $\hat{\mathcal{L}}/\hat{\sigma}$. Since the involution $\hat{\sigma}$ fixes no points, then any local coordinate at $\hat{P}$ serves as a local coordinate at $\pi_{\rm H}(\hat{P})$ [9], III.7.8. In particular, the branch points of $\hat{\mathcal{L}}$ map onto branch points of $\hat{\mathcal{L}}/\hat{\sigma}$. We conclude that $\mathcal{L}_{\rm H} = \hat{\mathcal{L}}/\hat{\sigma}$ is a hyperelliptic Riemann surface $\mathcal{L}_{\rm H}$ given by

$$\mu_{\rm H}^2 = (K-\xi)(K-\bar{\xi})\prod_{j=1}^{g}(K-E_j)(K-F_j). \tag{7.10}$$

To verify that this factorization really leads to $\mathcal{L}_{\rm H}$ we compute the genus γ of this surface using the Riemann-Hurwitz relation. The projection (7.9) is a holomorphic mapping of degree 2 with no branch points and we conclude that

$$\gamma = \frac{\hat{g}+1}{2} = g,$$

which is the genus of the hyperelliptic surface given by (7.10). The factorization of $\hat{\mathcal{L}}$ is diagrammatized in figure 4.

Now we can use the results of section 5.1 to construct a matrix function Ψ on $\mathcal{L}_{\rm H}$ and project it to $\mathcal{L}$ so it obeys the conditions of theorem 7.1. Although we know only a solution of the scalar Riemann-Hilbert problem, definition 5.1, we construct a matrix Ψ using this solution by exploiting its symmetry.

To fulfill the reality condition (iv) of theorem 7.1 we must impose more restrictions on the branch points of $\mathcal{L}_{\rm H}$. Namely, we request that each pair of the branch points satisfies either $E_j, F_j \in \mathbb{R}$ or $E_j = \bar{F}_j$. Then $\mathcal{L}_{\rm H}$ admits the antiholomorphic involution $\tau_{\rm H}$,

$$P_{\rm H} \equiv (K, \mu_{\rm H}(K)) \to \tau_{\rm H}(P_{\rm H}) \equiv \overline{P_{\rm H}} = (\overline{K}, \mu_{\rm H}(\overline{K})),$$

as well as the holomorphic involution $\sigma_{\rm H}$

$$P_{\rm H} \equiv (K, \mu_{\rm H}(K)) \to \sigma_{\rm H}(P_{\rm H}) = (K, -\mu_{\rm H}(K)).$$

On $\mathcal{L}_{\rm H}$ we use the cut system and the canonical homology basis indicated in figure 5. In the following, φ will be the Abelian mapping $\mathcal{L}_{\rm H} \to J(\mathcal{L}_{\rm H})$ with the base point $P_0 \equiv (\xi, 0)$ and θ will be the theta function on $\mathcal{L}_{\rm H}$. Let Ω be the abelian differential defined in section 5.1, satisfying the reality condition

$$\bar{\Omega}(P_{\rm H}) = \Omega(\overline{P_{\rm H}}), \tag{7.11}$$

D be the divisor of degree g that consists only of the branch points of $\mathcal{L}_{\rm H}$ and every point of D has multiplicity 1. Let Γ be a contour on $\mathcal{L}_{\rm H}$ subject to the reality condition

$$P_{\rm H} \in \Gamma \Rightarrow \overline{P_{\rm H}} \in \Gamma$$

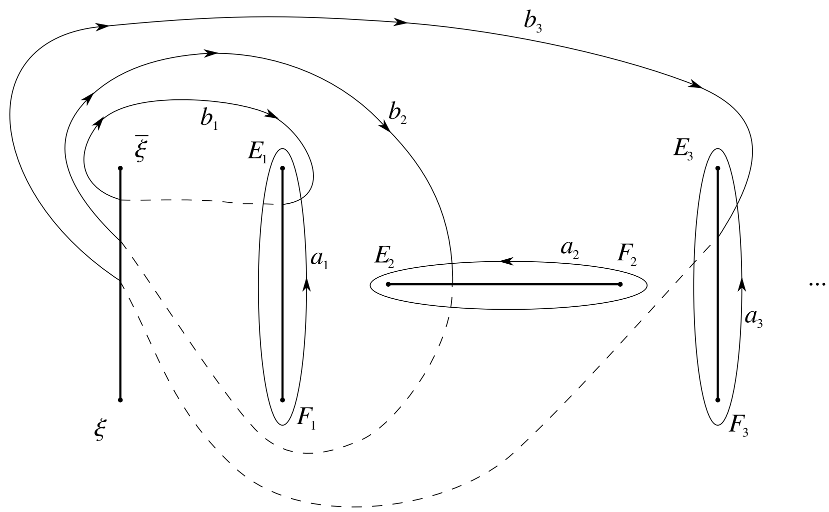


Figure 5: Cut system and canonical homology basis on $\mathcal{L}_{\mathrm{H}}$ viewed as a 2-sheeted cover of $\mathbb{CP}^1$. The dashed part of a cycle goes on the $-$-sheet of the cover.

and G be a Hölder continuous function of Γ satisfying

$$\overline{G(P_{\mathrm{H}})} = G(\overline{P_{\mathrm{H}}}), \quad P_{\mathrm{H}} \in \Gamma.$$

Let u and v be the vectors introduced in section 5.1. The function $\psi(P_{\mathrm{H}})$ (5.7) solves the scalar Riemann-Hilbert problem 5.1 on $\mathcal{L}_{\mathrm{H}}$. We define another function on $\mathcal{L}_{\mathrm{H}}$ by

$$\chi(P_{\mathrm{H}}) = \chi_0 \frac{\theta(\varphi(P_{\mathrm{H}}) - \varphi(\bar{P}_0) - \varphi(D) + u + v - \mathcal{K})}{\theta(\varphi(P_{\mathrm{H}}) - \varphi(D) - \mathcal{K})} \times \exp\left\{\Omega(P_{\mathrm{H}}) + \frac{1}{2\pi\mathrm{i}}\int_\Gamma \ln G\ \omega_{P_{\mathrm{H}}P_0}\right\}, \quad (7.12)$$

where χ_0 is a normalization constant. We will denote the argument of exponential by $I(P_{\mathrm{H}}, P_0)$,

$$I(P_{\mathrm{H}}, P_0) = \Omega(P_{\mathrm{H}}) + \frac{1}{2\pi\mathrm{i}}\int_\Gamma \ln G\ \omega_{P_{\mathrm{H}}P_0}.$$

The analytic behavior of $\chi(P_{\mathrm{H}})$ is identical to that of $\psi(P_{\mathrm{H}})$, except that it changes sign when it is continued along any of the cycles b_j. $\chi(P_{\mathrm{H}})$ is thus not a single-valued function on $\mathcal{L}_{\mathrm{H}}$ but it can be regarded as a function that changes sign at every cycle[20] a_j. However, it is single valued on $\hat{\mathcal{L}}$. $\hat{\mathcal{L}}$ can be viewed as two copies of $\mathcal{L}_{\mathrm{H}}$ glued in the following way: Two copies of $\mathcal{L}_{\mathrm{H}}$ cover the Riemann sphere $\mathbb{CP}^1$ by four sheets in total, with branch points projecting on the same points on $\mathbb{CP}^1$. We number the sheets of the first $\mathcal{L}_{\mathrm{H}}$ as 1 and 2 and the sheets of the second one as 3 and 4. The branch cuts interchange sheets 1, 2 and sheets 3, 4. We cut both surfaces along the loop above the branch cut $[\xi, \bar{\xi}]$ and glue the sheets in such a way that this branch cut interchanges the sheets 1, 3 and the sheets 2, 4. Then the lifts[21] of the cycles a_j from $\mathcal{L}_{\mathrm{H}}$ to $\hat{\mathcal{L}}$ divides $\hat{\mathcal{L}}$ into two disconnected components[22]. We will denote those two components $\hat{\mathcal{L}}^+$ and $\hat{\mathcal{L}}^-$ and they shall be fixed by

$$(\xi, 0, \pm\hat{\mu}(\xi)) \in \hat{\mathcal{L}}^{\pm}.$$

We normalize (if possible, see theorem 7.2) ψ and χ on $\mathcal{L}_{\mathrm{H}}$ such that

$$\psi(\infty_{\mathrm{H}}^-) = 1, \quad \chi(\infty_{\mathrm{H}}^-) = -1. \tag{7.13}$$

Now we will define functions $\hat{\psi}$ and $\hat{\chi}$ on $\hat{\mathcal{L}}$ by the relations

$$\begin{aligned} \hat{\psi}(\hat{P}) &= \psi(\pi_{\mathrm{H}}(\hat{P})), \\ \hat{\chi}(\hat{P}) &= \pm\chi(\pi_{\mathrm{H}}(\hat{P})), \quad \hat{P} \in \hat{\mathcal{L}}^{\pm}. \end{aligned} \tag{7.14}$$

Functions $\hat{\psi}$ and $\hat{\chi}$ defined this way are single-valued and holomorphic on $\hat{\mathcal{L}}$. We construct the matrix Ψ on $\mathcal{L}$ as

$$\Psi(P) = \begin{pmatrix} \hat{\psi}(P^{\oplus}) & \hat{\psi}(P^{\ominus}) \\ \hat{\chi}(P^{\oplus}) & \hat{\chi}(P^{\ominus}) \end{pmatrix}. \tag{7.15}$$

where $P^{\oplus}$ and $P^{\ominus}$ are the possible lifts of $P \equiv (K, \mu_0(K)) \in \mathcal{L}$ to $\hat{\mathcal{L}}$

$$P^{\oplus} = (K, \mu_0(K), +\hat{\mu}(K)), \quad P^{\ominus} = (K, \mu_0(K), -\hat{\mu}(K)).$$

The main result of this section is the following theorem:

Theorem 7.2. *Assume that* $\theta(\varphi(\infty_{\mathrm{H}}^-) - \varphi(D) - \mathcal{K}) \neq 0$. *Then the* 2×2 *matrix function* $\Psi(P)$ *on* $\mathcal{L}$ *defined in* (7.15) *satisfies the conditions of theorem 7.1.*

Proof. Verification of the reduction condition (iii) is straightforward. We have to check that the action of σ on Ψ is in accordance with (7.5). The

[20] Because the intersection number of any loop with a_j cycles counts the number of b_j cycles in this loop, see 3.2.

[21] The two components of the preimage of each cycle a_j under the projection π_{H}.

[22] The components of $\hat{\mathcal{L}} \setminus \cup_{j=1}^{g} {\pi_{\mathrm{H}}}^{-1}(a_j)$.

involution σ on $\mathcal{L}$ has a natural lift to $\hat{\mathcal{L}}$ that changes the sign in front of $\mu_0(K)$ only in $(K, \mu_0(K), \hat{\mu}(K))$. For a fixed point $P \in \mathcal{L}$, let $P_{\rm H} = \pi_{\rm H}(P^{\oplus})$. Then $\sigma_{\rm H}(P_{\rm H}) = \pi_{\rm H}(P^{\ominus})$ and the matrix (7.15) has the form

$$\Psi(P) = \begin{pmatrix} \psi(P_{\rm H}) & \psi(\sigma_{\rm H}(P_{\rm H})) \\ s_1\chi(P_{\rm H}) & s_2\chi(\sigma_{\rm H}(P_{\rm H})) \end{pmatrix},$$

with the signs s_1, s_2 fixed according to (7.14). If $P^{\oplus}$ and $P^{\ominus}$ lie in different components $\hat{\mathcal{L}}^{\pm}$ then $s_1 = -s_2$, if they lie in the same component then $s = s_1 = s_2$. Since $\pi_{\rm H}(\sigma(P)^{\oplus}) = \sigma_{\rm H}(\pi_{\rm H}(P^{\oplus}))$ and $\pi_{\rm H}(\sigma(P)^{\ominus}) = \sigma_{\rm H}(\pi_{\rm H}(P^{\ominus}))$ and because the involution σ changes the signs s_1, s_2 only when $P^{\oplus}$ and $P^{\ominus}$ lie in the same component, we get either

$$\Psi(P^{\sigma}) = \begin{pmatrix} \psi(\sigma_{\rm H}(P_{\rm H})) & \psi(P_{\rm H}) \\ s_1\chi(\sigma_{\rm H}(P_{\rm H})) & s_2\chi(P_{\rm H}) \end{pmatrix},$$

when the signs are different or

$$\Psi(P^{\sigma}) = \begin{pmatrix} \psi(\sigma_{\rm H}(P_{\rm H})) & \psi(P_{\rm H}) \\ -s\chi(\sigma_{\rm H}(P_{\rm H})) & -s\chi(P_{\rm H}) \end{pmatrix},$$

when the signs are the same and therefore we conclude that

$$\Psi(P^{\sigma}) = \sigma_3 \Psi(P) \sigma_1,$$

which is exactly the form of of the condition (iii).

Let $\mathcal{B} = \{E_1, F_1, \ldots, E_g, F_g\} \subset \mathbb{C}$ denote the set of projections of the constant branch points of $\mathcal{L}_{\rm H}$. By virtue of theorem 4.2 the functions ψ and χ have zeros on $\mathcal{L}_{\rm H}$ only at the points of the divisor D. These points lie only at the branch points $E_1, F_1, \ldots, E_g, F_g$ of $\mathcal{L}_{\rm H}$. At all other points the matrix function Ψ on $\mathcal{L}$ is therefore holomorphic and invertible. At a branch point, the situation is different. Let $P = (E, \mu_0(E)) \in \mathcal{L}$ and $E \in \mathbb{C}$ its projection on $\mathbb{CP}^1$, such that $E \in \mathcal{B}$. In a neighborhood of this point, $z = K - E$ is a local coordinate. The point $P_{\rm H} = (E, 0)$ is a branch point of $\mathcal{L}_{\rm H}$ then. In a neighborhood of this point, $z_{\rm H} = \sqrt{z} = \sqrt{K - E}$ is a local coordinate. The local coordinate at the point $\sigma_{\rm H}(P_{\rm H})$ is simply $z_{\rm H}(\sigma_{\rm H}(P_{\rm H})) = -z_{\rm H}(P_{\rm H})$. If $P_{\rm H} \in D$, functions χ and ψ have poles of order 1 at $P_{\rm H}$ and they can be written in the form

$$\begin{aligned} \psi(z_{\rm H}) &= \frac{a_{-1}}{z_{\rm H}} + a_0 + O(z_{\rm H}), \\ \chi(z_{\rm H}) &= \frac{b_{-1}}{z_{\rm H}} + b_0 + O(z_{\rm H}). \end{aligned}$$

The components of Ψ thus diverges as $(K-E)^{-1/2}$ at P and the determinant $\det\Psi$ has the form

$$\det\Psi(z) = \frac{2a_{-1}b_0 - 2a_0b_{-1}}{z_{\rm H}} + O(1)$$

and diverges as $(K-E)^{-1/2}$ as well. If $P_{\rm H} \notin D$, functions χ and ψ are regular at $P_{\rm H}$ and they can be written in the form

$$\begin{aligned}\psi(z_{\rm H}) &= a_0 + a_1 z_{\rm H} + O(z_{\rm H}^2),\\ \chi(z_{\rm H}) &= b_0 + b_1 z_{\rm H} + O(z_{\rm H}^2).\end{aligned}$$

The components of Ψ are thus regular at P and the determinant $\det\Psi$,

$$\det\Psi(z) = 2(a_1b_0 - a_0b_1)z_{\rm H} + O(z_{\rm H}^2),$$

vanishes as $(K-E)^{1/2}$. Because Ψ is a function on $\mathcal{L}_{\rm H}$ it is not single valued on $\mathcal{L}$. It has a jump across every lift of branch cuts $[E_j, F_j]$ to $\mathcal{L}$. If $E_j, F_j \in \mathbb{R}$, we get $\Psi_- = -\Psi_+\sigma_2$ on the cut between $[E_j, F_j]$, whereas we get $\Psi_- = -\Psi_+\sigma_1$ when $E_j = \overline{F_j}$ [21]. The logarithmic derivatives $\Psi_\xi\Psi^{-1}$ and $\Psi_{\bar\xi}\Psi^{-1}$ are, however, holomorphic at all of these points [21]. The condition (ii) is thus satisfied. At the points ξ and $\bar\xi$, the functions $\Psi_\xi\Psi^{-1}$ and $\Psi_{\bar\xi}\Psi^{-1}$ have poles of order 1, respectively, in accordance with (i) [20]. These properties of $\Psi_\xi\Psi^{-1}$ and $\Psi_{\bar\xi}\Psi^{-1}$ follow from the modular properties[23] of the theta function, but it is not pursued in our work.

To verify the condition (iv), i. e. that the matrix $\Psi(\infty^+)$ (7.15) has the form

$$\Psi(\infty^+) = \begin{pmatrix}\overline{\mathcal{E}} & 1\\ \mathcal{E} & -1\end{pmatrix},$$

we rewrite the functions ψ and χ first. We know that $\mathcal{K}$, $\varphi(D)$ and $\varphi(\overline{P_0})$ are points of order 2 in $J(\mathcal{L}_{\rm H})$ (see [9], VII.1). Therefore we can define the following integer characteristics. Let p, q, α and $\beta \in \mathbb{Z}^g$ be integer vectors such that

$$\begin{bmatrix}p\\ q\end{bmatrix} = \varphi(D) + \varphi(\overline{P_0}) + \mathcal{K}, \quad \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \varphi(D) + \mathcal{K}.$$

Since $\varphi(\overline{P_0}) \in \mathbb{R}$, it follows that $p = \alpha$. Then $\psi(P_{\rm H})$ and $\chi(P_{\rm H})$ can be

[23]Dependence of the theta function on the shape of a Riemann surface it resides on.

written using (4.4) in the form

$$\psi(P_{\mathrm{H}}) = \psi_0 \frac{\theta\begin{bmatrix}p\\q\end{bmatrix}(\varphi(P_{\mathrm{H}}) + u + v + \varphi(\overline{P_0}))}{\theta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\varphi(P_{\mathrm{H}}))} \exp \pi \mathrm{i}\left(p^T(u+v+\varphi(\overline{P_0})) + \frac{1}{2}p^T q - \frac{1}{2}\alpha^T\beta\right) \exp I(P_{\mathrm{H}}, P_0)$$

and similarly

$$\chi(P_{\mathrm{H}}) = \chi_0 \frac{\theta\begin{bmatrix}p\\q\end{bmatrix}(\varphi(P_{\mathrm{H}}) + u + v)}{\theta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\varphi(P_{\mathrm{H}}))} \exp \pi \mathrm{i}\left(p^T(u+v) + \frac{1}{2}p^T q - \frac{1}{2}\alpha^T\beta\right) \exp I(P_{\mathrm{H}}, P_0).$$

Since $\theta\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right](\varphi(\infty_{\mathrm{H}}^-) + u + v) \neq 0$, we can normalize ψ and χ according to (7.13). That yields for ψ at the point ∞_{H}^+

$$\psi(\infty_{\mathrm{H}}^+) = \frac{\theta\begin{bmatrix}p\\q\end{bmatrix}(\varphi(\infty_{\mathrm{H}}^+) + u + v + \varphi(\overline{P_0}))}{\theta\begin{bmatrix}p\\q\end{bmatrix}(\varphi(\infty_{\mathrm{H}}^-) + u + v + \varphi(\overline{P_0}))} \frac{\theta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\varphi(\infty_{\mathrm{H}}^-))}{\theta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\varphi(\infty_{\mathrm{H}}^+))} \exp I(\infty_{\mathrm{H}}^+, \infty_{\mathrm{H}}^-)$$

and similarly for $\chi(\infty_{\mathrm{H}}^+)$. Because the lifts of ∞_{H}^+ and ∞_{H}^- lie in different components $\hat{\mathcal{L}}^{\pm}$, the sign of $\chi(\infty_{\mathrm{H}}^+)$ changes and thus the expressions for $\psi(\infty_{\mathrm{H}}^+)$ and $\chi(\infty_{\mathrm{H}}^+)$ differ only in the argument of the theta functions in the first fraction. The second fraction is 1, because $\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$ is an even characteristic (see [9], VII.1) and $\varphi(\infty_{\mathrm{H}}^+) = -\varphi(\infty_{\mathrm{H}}^-)$ since the integration path in φ has the same projection on $\mathbb{CP}^1$.

We have to show that for the normalized functions ψ and χ the relation

$$\overline{\psi(\infty_{\mathrm{H}}^+)} = \chi(\infty_{\mathrm{H}}^+)$$

holds. We will need the following lemma.

Lemma 7.3 (Reality properties)**.** *Let $\mathcal{L}_{\mathrm{H}}$ be a hyperelliptic suface of genus g with branch points that have projections $\xi, \bar{\xi}, E_1, F_1, \ldots, E_g, F_g$ on $\mathbb{CP}^1$ that satisfies the following reality condition: For each pair of branch points E_j, F_j,*

$j = 1, \dots, g$, it is either $E_j, F_j \in \mathbb{R}$ or $E_j = \overline{F_j}$. Let the canonical homology basis be given as shown in figure 5. Then

$$\varphi(\overline{P_0}) = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$$

as a point in the Jacobian variety $J(\mathcal{L}_\mathrm{H})$. Furthermore, the matrix Π of b-periods has the reality property

$$\Pi + \overline{\Pi} = C,$$

where

$$C = \begin{pmatrix} d_1 & 1 & \cdots & 1 \\ 1 & d_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & d_g \end{pmatrix}, \quad d_j = \begin{cases} 1 & \text{if } E_j, F_j \in \mathbb{R} \\ 0 & \text{if } E_j = \overline{F_j} \end{cases}, \quad j = 1, \dots, g.$$

Proof. We know that σ_H acts on the differentials of the normalized basis ω by $\sigma_\mathrm{H}(\omega) = -\omega$. Let a_0 be a cycle running around the branch points $\xi, \bar{\xi}$ on the $+$-sheet of $\mathcal{L}_\mathrm{H}$ in the counter-clockwise direction. It can be decomposed in the canonical homology basis using (3.5), $a_0 = -\sum_{j=1}^{g} a_j$. Then

$$\int_{\xi}^{\bar{\xi}} \omega = \frac{1}{2} \int_{a_0} \omega = \frac{1}{2}(-1, \dots, -1) = \frac{1}{2}(1, \dots, 1).$$

The reality property of the matrix Π can be established by employing the antiholomorphic involution τ_H. We express $\tau_\mathrm{H}(a_j)$ and $\tau_\mathrm{H}(b_j)$, which is just another canonical homology basis[24], in the canonical homology basis a_j, b_j using (3.5) and we obtain

$$\tau_\mathrm{H}(a_j) = -a_j, \quad \tau_\mathrm{H}(b_j) = b_j - \sum_{\substack{k=1 \\ k \neq j}}^{g} a_k - \begin{cases} a_j & \text{if } E_j, F_j \in \mathbb{R} \\ 0 & \text{if } E_j = \overline{F_j} \end{cases}.$$

We see that

$$\begin{pmatrix} \tau_\mathrm{H}(a) \\ \tau_\mathrm{H}(b) \end{pmatrix} = \begin{pmatrix} -I & 0 \\ I & -C \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and we immediately obtain

$$\Pi + \overline{\Pi} = C$$

from (3.11). □

[24] Even though $\tau_\mathrm{H}(a_j) \cdot \tau_\mathrm{H}(b_j) = -1$, we can call it a canonical homology basis for our purposes

Corollary 7.4. *Let $\theta(z)$ be the theta function on $\mathcal{L}_{\mathrm{H}}$ with reality properties given by the above lemma. Then*

$$\overline{\theta(z)} = \theta(\bar{z} + \frac{1}{2}d). \tag{7.16}$$

Proof. By expressing $\overline{\theta(z,\Pi)}$ into the defining series

$$\overline{\theta(z,\Pi)} = \sum_{N\in\mathbb{Z}^g} \exp 2\pi\mathrm{i}\left(\frac{1}{2}N^T\Pi N - N^T\bar{z} - \frac{1}{2}N^TCN\right)$$

we observe that

$$\pi\mathrm{i}N^TCN = \pi\mathrm{i}\sum_{j=1}^{g} N_i^2 d_i + 2\pi\mathrm{i}\sum_{\substack{j=1\\k=1\\j\neq k}}^{g} N_j N_k.$$

The first term gives in the exponential the same result as N^Td, while the second term gives 1, and we immediately obtain (7.16). □

Now we have

$$-\varphi(\infty_{\mathrm{H}}^{+}) - \varphi(\overline{P_0}) - u - v - \frac{1}{2}d = \overline{\varphi(\infty_{\mathrm{H}}^{+}) + u + v}$$

and we conclude that the matrix $\Psi(P)$ obeys the normalization condition (iv) as well. □

At this moment we have everything we need to write the formula for an Ernst potential from the constructed matrix $\Psi(P)$.

Corollary 7.5 (Ernst potential). *Let $\theta\left[{p \atop q}\right](\varphi(\infty_{\mathrm{H}}^{-}) + u + v) \neq 0$. Then the function*

$$\mathcal{E}(\xi,\bar{\xi}) = \frac{\theta\begin{bmatrix}p\\q\end{bmatrix}(\varphi(\infty_{\mathrm{H}}^{+}) + u + v)}{\theta\begin{bmatrix}p\\q\end{bmatrix}(\varphi(\infty_{\mathrm{H}}^{-}) + u + v)} \times \exp\left(\Omega(\infty_{\mathrm{H}}^{+}) - \Omega(\infty_{\mathrm{H}}^{-}) + \frac{1}{2\pi\mathrm{i}}\int_\Gamma \ln G\ \omega_{\infty^+\infty^-}\right).$$

is a solution to the Ernst equation.

Proof. We know that the matrix $\Psi(P)$ is subject to the conditions of theorem 7.1 and therefore we obtain the Ernst potential from $\Psi(\infty^+)$ directly. □

The solutions studied here are a subclass of solutions found by Korotkin [23]. pq has to be an integer characteristic.

7.3 Symmetric linear system

There is another useful (more symmetric) form of the linear system (7.1) (see [27]). Introducing the real symmetric matrix

$$g(\xi,\bar{\xi}) = \begin{pmatrix} 2 & \mathrm{i}(\mathcal{E}-\bar{\mathcal{E}}) \\ \mathrm{i}(\mathcal{E}-\bar{\mathcal{E}}) & 2\mathcal{E}\bar{\mathcal{E}} \end{pmatrix}, \tag{7.17}$$

the Ernst equation can be rewritten as

$$((\xi-\bar{\xi})g_\xi g^{-1})_{\bar{\xi}} + ((\xi-\bar{\xi})g_{\bar{\xi}} g^{-1})_\xi = 0. \tag{7.18}$$

This equation is the compatibility condition of the linear system

$$\Psi_\xi = \frac{g_\xi g^{-1}}{1-\gamma}\Psi, \qquad \Psi_{\bar{\xi}} = \frac{g_{\bar{\xi}} g^{-1}}{1+\gamma}\Psi, \tag{7.19}$$

where $\Psi(\gamma,\xi,\bar{\xi})$ is a 2×2 matrix function with continuous second derivatives. The function $\gamma(\xi,\bar{\xi})$ is a spectral parameter subject to the compatible first order equations

$$\gamma_\xi = \frac{\gamma}{\xi-\bar{\xi}}\frac{1+\gamma}{1-\gamma}, \qquad \gamma_{\bar{\xi}} = \frac{\gamma}{\bar{\xi}-\xi}\frac{1-\gamma}{1+\gamma}.$$

They can be solved by

$$\gamma_\pm(w,\xi,\bar{\xi}) = \frac{2}{\xi-\bar{\xi}}\left[w - \frac{\xi+\bar{\xi}}{2} \pm \sqrt{(w-\xi)(w-\bar{\xi})}\right] = \frac{1}{\gamma_\mp(w,\xi,\bar{\xi})}$$

with $w \in \mathbb{C}$ a constant of integration, which can be regarded as a hidden spectral parameter. In the sequel we will suppress the index $\pm$ and simply write $\gamma(w,\xi,\bar{\xi}) \equiv \gamma_\pm(w,\xi,\bar{\xi})$. It can be regarded as a meromorphic function of degree 1 on the Riemann surface $\mathcal{L}$ of genus 0, given by the relation $\mu_{\mathrm{w}}^2 = (w-\xi)(w-\bar{\xi})$. This function is thus a holomorphic bijection $\mathcal{L} \to \mathbb{CP}^1$. It has a pole of order 1 at ∞^+ and zero of order 1 at ∞^-. We can describe all quantities either on $\mathcal{L}$ as functions of $P = (w, \mu_{\mathrm{w}}(w)) \in \mathcal{L}$, or on $\mathbb{CP}^1$ as functions of $\gamma \in \mathbb{CP}^1$. We will adhere to the latter approach here. Furthermore, using the chain rule we get

$$\frac{d}{d\xi} \equiv \frac{\partial}{\partial\xi} + \frac{\gamma}{\xi-\bar{\xi}}\frac{1+\gamma}{1-\gamma}\frac{\partial}{\partial\gamma}, \qquad \frac{d}{d\bar{\xi}} \equiv \frac{\partial}{\partial\bar{\xi}} + \frac{\gamma}{\bar{\xi}-\xi}\frac{1-\gamma}{1+\gamma}\frac{\partial}{\partial\gamma}. \tag{7.20}$$

As before, we investigate the behavior of $\Psi_\xi\Psi^{-1}$ and $\Psi_{\bar{\xi}}\Psi^{-1}$ as that of functions of γ on the Riemann sphere $\mathbb{CP}^1$. The following theorem summarizes the analytic properties of Ψ as a function of γ [27].

Theorem 7.6 (Analytic properties of $\Psi(\gamma)$). *Let the* 2×2 *matrix* $\Psi(\gamma, \xi, \bar{\xi})$ *be subject to the following conditions:*

(*i*). *As a function of* γ *the matrix* Ψ *is holomorphic and invertible everywhere on some cover of the Riemann* γ*-sphere* $\mathbb{CP}^1$ *with the exception of the points mentioned below.*

(*ii*). Ψ *has regular singularities at the branch points* $\gamma_j(\xi, \bar{\xi}) = \gamma(w_j, \xi, \bar{\xi})$ *for* $j = 1, \ldots, N$ *with constants* $w_j \in \mathbb{C}$, *in the vicinity of which it behaves as*

$$\Psi(\gamma, \xi, \bar{\xi}) = G_j(\xi, \bar{\xi})\Psi_j(\gamma, \xi, \bar{\xi})(\gamma - \gamma_j)^{T_j} C_j \quad as\ \gamma \sim \gamma_j,$$

where, for $\gamma \sim \gamma_j$, $\Psi_j(\gamma, \xi, \bar{\xi}) = I + O(\gamma - \gamma_j)$ *is holomorphic and invertible . The matrices* C_j *and* T_j *are constant invertible and constant diagonal, respectively, while* ξ, $\bar{\xi}$ *dependent matrices* G_j *are assumed to be invertible.*

(*iii*). *Across certain contours* $\{L_j\}$, $L_j \subset \mathbb{CP}^1$, $L_j = L_j(\xi, \bar{\xi})$, *which connect the singular points* γ_j *to some arbitrarily choosen fixed and non-singular base point* $\gamma_0 = \gamma(w_0, \xi, \bar{\xi})$, *the boundary values of* $\Psi_-(\gamma)$ *and* $\Psi_+(\gamma)$ *are related by*

$$\Psi_+(\gamma, \xi, \bar{\xi}) = \Psi_-(\gamma, \xi, \bar{\xi})M_j(w), \qquad \gamma \in L_j,$$

where the invertible matrices M_j *depend only on the constant spectral parameter* w.

(*iv*). *The normalization conditions*

$$\Psi(\infty, \xi, \bar{\xi}) = g_\infty, \tag{7.21}$$

$$\Psi(0, \xi, \bar{\xi}) = g(\xi, \bar{\xi}) \tag{7.22}$$

hold, where the matrix g_∞ *is constant invertible and the matrix* $g(\xi, \bar{\xi})$ *is invertible.*

Then Ψ *obeys the linear system* (7.19) *and* $g(\xi, \bar{\xi})$ *solves* (7.18).

We present a more detailed proof here, exploiting the Riemann-Rochh theorem again.

Proof. Conditions (i) – (iii) imply that $\Psi_\xi \Psi^{-1}$ is holomorphic in γ everywhere on $\mathbb{CP}^1$ away from $\gamma = 1$. Indeed, it has removable singularities at singularities of Ψ. If $\gamma \sim \gamma_j$, then from (7.20)

$$\frac{d}{d\xi}(\gamma - \gamma_j)^{T_j} = T_j(\gamma - \gamma_j)^{T_j - I}\frac{(\gamma - \gamma_j)(1 + \gamma + \gamma_j - \gamma\gamma_j)}{(1 - \gamma)(1 - \gamma_j)} = S_j(\gamma - \gamma_j)^{T_j}$$

with S_j holomorphic, and thus the singular terms $(\gamma-\gamma_j)^{T_j}$ and $(\gamma-\gamma_j)^{-T_j}$ in $\Psi_\xi\Psi^{-1}$ cancel each other at $\gamma\sim\gamma_j$. At the contours L_j the situation is even simpler because

$$\frac{d\Psi_+}{d\xi}{\Psi_+}^{-1}=\frac{d\Psi_-}{d\xi}M_j(w){M_j}^{-1}(w){\Psi_-}^{-1}=\frac{d\Psi_-}{d\xi}{\Psi_-}^{-1}.$$

By virtue of (7.20), it has a pole of order 1 at $\gamma=1$, with behavior

$$\Psi_\xi\Psi^{-1}=\Psi_\gamma\Psi^{-1}\frac{2}{\xi-\bar{\xi}}\frac{1}{1-\gamma}+O(1),\quad \gamma\sim 1.$$

Now we use the same argument as in theorem 7.1. $\Psi_\xi\Psi^{-1}$ is a meromorphic function of degree 1 with a pole at $\gamma=1$. Hence, it must be of the form

$$\Psi_\xi\Psi^{-1}=\beta_0\frac{1}{1-\gamma}+\beta_1,$$

where β_0 and β_1 are constant (independent of γ, but depending on ξ and $\bar{\xi}$) 2×2 matrices. Both are fixed by the normalization (iv), $\beta_1=0$ is fixed by (7.21) and $\beta_0=g_\xi g^{-1}$ is fixed by (7.22). We conclude that $\Psi_\xi\Psi^{-1}$ has the form of the first equation in (7.19). The form of the equation for $\Psi_{\bar{\xi}}\Psi^{-1}$ is obtained in the same way, just with the pole at $\gamma=-1$. □

A function $\Psi(\gamma,\xi,\bar{\xi})$ can be constructed using a solution to a Riemann-Hilbert problem with quasi-permutation monodromy matrices, see section 5.2. To make sure that $g(\xi,\bar{\xi})$ obeys not only the equation 7.18, but also is real and symmetric and an Ernst potential can be reconstucted from it, one has to impose further conditions on the function $\Psi(\gamma,\xi,\bar{\xi})$, see [27]. The Ernst potential obtained in this way can be rewritten in the form of the Ernst potential found in section 7.2, see for example [18, 26]. The advantage of this more general approach will be probably a possibility to construct a solution to the Einstein-Maxwell equations in stationary axisymmetric case, see [18].

8 Numerical evaluation of the theta solutions

The solutions of the Ernst equation in terms of theta functions on hyperelliptic Riemann surfaces have been known since the end of 1980s. Up to now, only a few papers have addressed their numerical evaluation. However, a fast and reliable method for obtaining numerical values of metric coefficient is a key condition of applicability of the solutions in astrophysics.

There are general packages from Deconinck et al. for computing values of general theta functions (see [6]) and parameters of general algebraic curves (see [5]) that are distributed with Maple 8. Other examples are approaches via Schottky uniformization [1]. These methods work well for the algebro-geometric solutions of evolution equations like Korteweg-de Vries, Sine-Gordon, or nonlinear Schrödinger equation. Unfortunately, their direct use for the Ernst-equation solutions is misleading. Namely, for each spacetime point the solution is implicitly parametrized by the underlying structure of the hyperelliptic Riemann surface it resides on. A change of spacetime coordinates moves two branch points of the surface and thus deforms the latter. It is therefore necessary to compute all the essential quantities characterizing the surface at every point of the ρ-ζ half-plane, especially the b-period matrix Π. To obtain those values, one has to integrate differentials on the surface, which leads to hyperelliptic integrals where the analytical continuation of functions is imperative. Analyzing the latter is time expensive for general Riemann surfaces. However, in the case of hyperelliptic surfaces it can be done more quickly, because one can exploit the symmetry of such a surface which can be regarded as a 2-sheeted cover of $\mathbb{CP}^1$. This was noticed by Frauendiener and Klein [10–12] who were able to handle numerically a special case of solutions of genus 2 with a thin disk source. Here we present and discuss an implementation of spectral methods used in the latter articles to hyperelliptic surfaces of general genus in Mathematica 5.0 and then we study the computed Ernst potential and metric functions.

We start from the property of a hyperelliptic surface $\mathcal{L}_{\mathrm{H}}$ of genus g that it can be given by simple formula (3.15) that in the case of the Ernst equation is parametrized by the physical coordinates $\xi, \bar{\xi}$ as

$$\mu^2 = (K - \xi)(K - \bar{\xi}) \prod_{i=1}^{g} (K - E_i)(K - F_i), \tag{8.1}$$

where K is the spectral parameter. It is therefore a 2-sheeted cover of the Riemann sphere $\mathbb{CP}^1$. We denote by ∞^+, ∞^- the points that project on infinity of the sphere and which are distinguished by the condition (3.16).

The basis of holomorphic differentials is simply given by

$$\nu_j = \frac{K^{j-1}}{\mu} dK, \qquad j = 1, \ldots, g. \tag{8.2}$$

The normalized differentials ω_j are their linear combinations. We define the normalized Abelian differential of the third kind $\omega_{\infty^+\infty^-}$ with poles at ∞^+ and ∞^- and with residues $+1$ and -1, respectively. It is normalized by the condition that all a-periods vanish, i. e. $\int_{a_j} \omega_{\infty^+\infty^-} = 0$ for $j = 1, \ldots, g$. It is given by the formula $\nu_{g+1} = -K^g dK/\mu$ up to a holomorphic differential. By virtue of the bilinear relations (see [9], III.3.6), the useful formula

$$\frac{1}{2\pi \mathrm{i}} \int_{b_k} \omega_{\infty^+\infty^-} = \varphi(\infty^+) - \varphi(\infty^-)$$

holds, where φ is the Abelian mapping (sect. 3.6). The solutions of the Ernst equation can be written in the form [19]

$$\mathcal{E} = \frac{\theta_{pq}(\varphi(\infty^+))}{\theta_{pq}(\varphi(\infty^-)))}. \tag{8.3}$$

Paths of integration in $\varphi(\infty^+)$ and $\varphi(\infty^-)$ have the same projection on the Riemann sphere. The constant vectors p and q must satisfy the reality condition $Bp + q \in \mathbb{R}^g$.

For large values of $|\xi|$ the Ernst potential expands as [11]

$$\mathcal{E} = 1 - \frac{2m}{|\xi|} - \frac{2m^2}{|\xi|^2} - \frac{2\mathrm{i}J\zeta}{|\xi|^3} + O\left(\frac{1}{|\xi|^3}\right) \tag{8.4}$$

the constants (with respect to ξ) m and J being the mass and the angular momentum of the spacetime, respectively. The mass m can in principle be imaginary. The real part of m is the ADM-mass (Arnowitt-Deser-Misner, [17]) and the imaginary part of m is called magnetic mass[25].

8.1 Numerics on a hyperelliptic Riemann surface

The underlying hyperelliptic Riemann surface $\mathcal{L}_\mathrm{H}$ parametrizes the solution (8.3) by the matrix of b-periods Π and by the values of $\varphi(\infty^\pm)$. To evaluate those quantities, we need to integrate holomorphic differentials (8.2) and the Abelian differential of the third kind $\omega_{\infty^+\infty^-}$ along the cycles $a_1, \ldots, a_g$,

[25] It is connected to the NUT (Newman-Unti-Tamburino, [17]) parameter β_0 by the relation $\arctan \beta_0 = \operatorname{Im} m / \operatorname{Re} m$

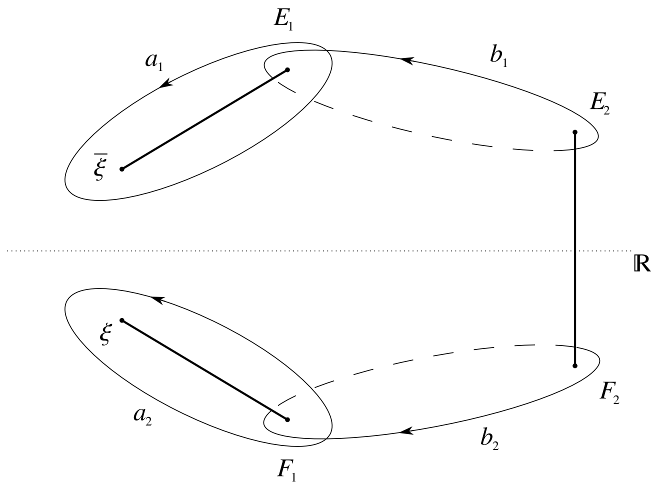


Figure 6: Cut system and the basis of cycles for the numerical treatment of a surface of genus 2

$b_1, \ldots, b_g$ of the fundamental group $\pi_1(\mathcal{L}_{\mathrm{H}})$. Since $\mathcal{L}_{\mathrm{H}}$ is the 2-sheeted cover of the sphere, we can evaluate those integrals on $\mathcal{L}_{\mathrm{H}}$ as integrals on the Riemann sphere with careful tracing of the sheet. We will denote the 2 sheets as $+$ and $-$ according to the point $\infty^{\pm}$ that they contain. Since the integral of a differential along the boundary of a simply connected domain vanishes provided that the differential is holomorphic inside the domain (Cauchy theorem), we can always continuously deform the cycle without changing the value of the integral. By deforming the cycle we can always express the integral along the cycle as a sum of line integrals of the type

$$\int_E^F \frac{K^j}{\sqrt{\mu^2}} dK, \qquad j = 0, \ldots, g \tag{8.5}$$

between some branch points E and F. Evaluation of such integrals is thus the main challenge of numerical treatment of the Ernst potential.

8.1.1 Square root

We need to track the sheet of the cover as effectively as possible. The crucial factor is the value of $\mu(K) = \sqrt{\mu^2(K)}$ defined by (8.1). The differentials also depend on μ so the evaluation of the square root $\sqrt{\mu^2(K)}$ for arbitrary complex number K is the essential part of the computation. In order to make this a well defined problem, we introduce the cut-system as shown in figure 6 for a surface of genus 2, or as shown in figure 5. On the cut surface the square root $\mu(K)$ is defined as the product of square roots of monomials

$$\mu = \sqrt{K-\xi}\sqrt{K-\bar{\xi}}\prod_{i=1}^{g}\sqrt{K-E_i}\sqrt{K-F_i}. \tag{8.6}$$

We cannot simply use the square root routines such as the one available in Mathematica. Namely, the expression (8.6) is holomorphic on the cut surface, but the built-in square root function usually has its branch-cut along the negative real axis. We need to adapt the square roots to out cut system.

Let E, F be two branch points connected by a branch-cut. Let $\alpha = \arg(F-E)$, where $\arg(z)$ is the argument of a complex number z with values $(-\pi, \pi)$. Now we define the square root $\sqrt[(\alpha)]{\cdot}$ with branch-cut along the ray with argument α by computing, for each $z \in \mathbb{C}$, the square root $s = \sqrt{z}$ with the help of the built-in routine and then putting

$$\sqrt[(\alpha)]{z} = \begin{cases} s & \alpha/2 < \arg(s) < \alpha/2 + \pi \\ -s & \text{otherwise} \end{cases}.$$

With the new square root $\sqrt[(\alpha)]{\cdot}$, we compute two factors in (8.6) as

$$\sqrt[(\alpha)]{K-E}\,\sqrt[(\alpha)]{K-F}.$$

The expression changes sign exactly on the branch-cut between E and F. The whole expression (8.6) is computed by multiplying the pairs of factors with square roots $\sqrt[(\alpha)]{\cdot}$ corresponding to the branch-cuts.

In the case of a nonlinear substitution to the integrals we will use later, however, the definition of the square root $\sqrt[(\alpha)]{\cdot}$ is useless. We have to continue the square root analytically along the path of integration. We use the following simple method. We assume that the argument of the square root is a complex-valued continuous function f of one real parameter $t \in [a, b]$. In our case it will be $f(t) = \mu^2(K(t))$. Therefore, the image $f([t_1, t_2])$ of the function is a connected curve in the complex plane. Furthermore, we assume that the square root is evaluated for a set of consecutive values $f(t_n)$, $a = t_0 \leq t_2 \leq \cdots \leq t_N = b$. Since the built-in square root has a branch-cut on the negative real axis, we test if the line between consecutive arguments $f(t_{i-1})$ and $f(t_i)$ of the square root in the complex plane crosses the negative real axis. If yes, we multiply the new value of the square root by -1 and then we alter all subsequent values of the square root in this way until the negative real axis is crossed again. The branch of the first value $\sqrt{f(t_0)}$ has to be chosen appropriately and we use the value given by (8.6).

8.1.2 Numerical integration of the periods

The task of following the analytical continuation of the functions $K^j/\mu(K)$ in (8.5) is much simplified by the proper definition of square roots in (8.6). However, the numerical integration of the line integrals (8.5) is still tricky. The integrands have singularities of the type $\frac{1}{\sqrt{K-E}}$ at the end points of the line of integration. On the other hand, the integrand is analytic in between. Using the linear parametrization $K = \frac{F-E}{2}(t+1) + E$ for a real parameter $t \in (-1, 1)$, we get integrals of the form

$$I = \int_{-1}^{1} \frac{P(t)H(t)}{\sqrt{1-t^2}} dt, \tag{8.7}$$

where $P(t) = K^j(t)$ is a polynomial of degree j and $H(t)$ is an analytic function of the form

$$H(t) = \frac{F-E}{\sqrt[(\alpha)]{F-E}\,\sqrt[(\beta)]{E-F}} \frac{\sqrt[(\alpha)]{K(t)-E}\,\sqrt[(\beta)]{K(t)-F}}{\mu(K(t))},$$

α and β being arguments of the branch-cuts corresponding to the branch points E and F. This form of the integral suggests to use spectral methods. To evaluate the integral, we approximate the function $H(t)$ by a linear combination of Chebyshev polynomials (see below)

$$H(t) = \sum_{n=0}^{N} h_n T_n(t).$$

Then we express the polynomial $P(t)$ in the basis of Chebyshev polynomials up to order j,

$$P(t) = \sum_{n=0}^{j} p_n T_n(t).$$

The integral I is then calculated with the help of the orthogonality properties of Chebyshev polynomials as

$$I = \pi p_0 h_0 + \frac{\pi}{2} \sum_{n=1}^{j} p_n h_n. \tag{8.8}$$

The Chebyshev polynomials $T_n(x)$ are defined on the interval $[-1, 1]$ by the formula

$$T_n(\cos(t)) = \cos(nt), \quad \text{where } x = \cos(t), \quad t \in [0, \pi].$$

They form an ortogonal system on $[-1,1]$ with respect to the inner product

$$\langle f,g\rangle \equiv \int_{-1}^{1}\frac{f(x)\bar{g}(x)}{\sqrt{1-x^2}}dx.$$

The normalization is

$$\langle T_m,T_n\rangle = c_m\frac{\pi}{2}\delta_{mn},$$

where $c_0 = 2$ and $c_m = 1$ otherwise. In order to approximate a continuous function $f \in C([-1,1])$ by a series of Chebyshev polynomials $\sum_{n=0}^{N} a_n T_n$ for given N, we require that

$$f_l \equiv f(x_l) = \sum_{n=0}^{N} a_n T_n(x_l) \quad \text{at points } x_l = \cos\left(\frac{\pi(l+\frac{1}{2})}{N+1}\right), \tag{8.9}$$

where $l = 0,\ldots,N$. By putting $c_0 = 2$ and $c_n = 1$ for $n = 2,\ldots,N$ and defining the numbers $F_n = c_n a_n$, we get

$$\begin{aligned} f_l &= \sum_{n=0}^{N} a_n T_n(x_l) = \sum_{n=0}^{N} a_n T_n\left(\cos\left(\frac{\pi(l+\frac{1}{2})}{N+1}\right)\right) \\ &= \sum_{n=0}^{N} a_n \cos\left(\frac{\pi n(l+\frac{1}{2})}{N+1}\right) = \sum_{n=0}^{N}\frac{F_n}{c_n}\cos\left(\frac{\pi n(l+\frac{1}{2})}{N+1}\right). \end{aligned}$$

This is exactly the discrete cosine series up to a constant. The coefficients F_n are therefore related to the values f_l of the function by the inverse discrete cosine transform (DCT) [35]

$$F_n = \frac{2}{N+1}\sum_{l=0}^{N} f_l \cos\left(\frac{\pi n(l+\frac{1}{2})}{N+1}\right).$$

The inverse discrete cosine transform is related to the discrete Fourier transform of the extended data f_l, $l = 0,\ldots,2N+1$, extended symmetrically around $n = N+1/2$, by [35]

$$f_{2N+1-j} = f_j, \qquad j = 0,\ldots,N.$$

The discrete cosine transform consists then of the first $N+1$ values of the discrete Fourier transform divided by the constants c_n and multiplied by the factor $e^{\pi i n/(2N+2)}$,

$$c_n F_n = \frac{2}{N+1}e^{\pi i n/(2N+2)}\sum_{l=0}^{2N+1} f_l e^{\pi i n l/(N+1)}, \quad n = 0,\ldots,N.$$

The discrete Fourier transform can be computed very effectively by the fast Fourier transform (FFT) routine[26]. This relationship between Chebyshev polynomials and the FFT is the basis for efficient computations.

A different set of test points x_l was used in [11,12], namely $x_l = \cos(\pi l/N)$. The advantage of our approach is that the test points (8.9) are the roots of the polynomial T_{n+1} which helps to estimate an error of the approximation (8.10). According to our checks, both methods perform equally well.

The approximation by Chebyshev polynomials of a function $H(t)$ can be used to compute the integral $\int_{-1}^{1} H(t)dt$. The formula known as the Clenshaw-Curtis quadrature [35], chapt. 5.9, reads

$$\int_{-1}^{1} H(t)dt = -2\sum_{k=0}^{[N/2]} \frac{h_{2k}}{(2k-1)(2k+1)},$$

where $[N/2]$ is the integer part of $N/2$. The Clenshaw-Curtis quadrature with the test points $x_l = \cos(\pi l/N)$, used in [11,12], is called the "trapezoidal" or Gauss-Lobato variant.

Let us remark that a polynomial $K^j(t)$ can be quickly expressed in the basis of Chebyshev polynomials up to order j by DCT with $j+1$ sample points $x'_l = \cos \pi l/j$.

The Chebyshev polynomial approximation is almost as good as the 'holy grail' of approximating polynomials, the *minimax polynomial* [35], but it has one serious drawback in our application. The estimation of the residue of the Chebyshev polynomial approximation is

$$\max_{-1\leq t\leq 1} |H(t) - \sum_{n=0}^{N} h_n T_n(t)| \leq \frac{1}{2^n(n+1)!} \max_{-1\leq t\leq 1} |H^{(n+1)}(t)|. \tag{8.10}$$

So the approximation does a great job when approximating smooth function with bounded derivations. In our case, however, the function $H(t)$ contains terms of the form $1/\sqrt{t-t_0}$. The n-th derivation of this term is

$$\frac{d}{dt^n}\frac{1}{\sqrt{t-t_0}} = (-1)^n \frac{(2n-1)!!}{2^n}(t-t_0)^{-\frac{2n+1}{2}},$$

where $(2n-1)!! = n(n-2)(n-4)\cdots$ is the double factorial (product of odd numbers not greater that $2n-1$). When $|t-t_0|$ is small, the error estimate (8.10) is even increasing with increasing N. Thus we do not get a

[26]There also exists a fast cosine transform, but special packages for its implementation are usually needed.

better approximation with larger N and the applicability of the Chebyshev approximation for the integration of (8.7) is very limited.

In our case problems occur when some other branch points are "too close" to the path of integration in (8.5), i. e. to the line connecting branch points E and F. "Too close" means that the numerical error of the Chebyshev approximation is too high and cannot be reduced by increasing the number of approximating polynomials. In some cases this problem can be avoided by a different choice of integration paths (8.5), especially when the branch points are distributed uniformly. But, unfortunately, there are situations when some branch points are very close to each other; in particular this applies to the spacetime symmetry axis when $\xi \to \bar{\xi}$, to radial infinity when $\xi, \bar{\xi} \to \infty$ and to the other branch points[27] when $\xi \to F_i$ while $\bar{\xi} \to E_i$.

To solve this problem, we will use the substitution of the form

$$K(t) = a \sin(ct + d) + b, \quad a, b, c, d \in \mathbb{C}, \quad t \in (-1, 1) \subset \mathbb{R} \tag{8.11}$$

in (8.5). We can regularize then the integrand

$$\frac{K^j}{\sqrt{(K - R)(K - S) \cdots}}$$

at exactly two singular points $R, S \in \mathbb{C}$ by a particular choice of constants $a = (S - R)/2$ and $b = (S + R)/2$. That means that the integral (8.5) is put into the form

$$\int_{t_1}^{t_2} c \frac{K^j(t)(S - R)\cos(ct + d)dt}{\sqrt{(S - R)(1 + \sin(ct + d))(R - S)(1 - \sin(ct + d)) \cdots}} \tag{8.12}$$

and the terms $\cos(ct + d)$ and $\sqrt{1 - \sin^2(ct + d)}$ cancel out with the result ± 1. The sign has to be fixed so that we integrate the correct branch of the square root. With the particular choice of constants

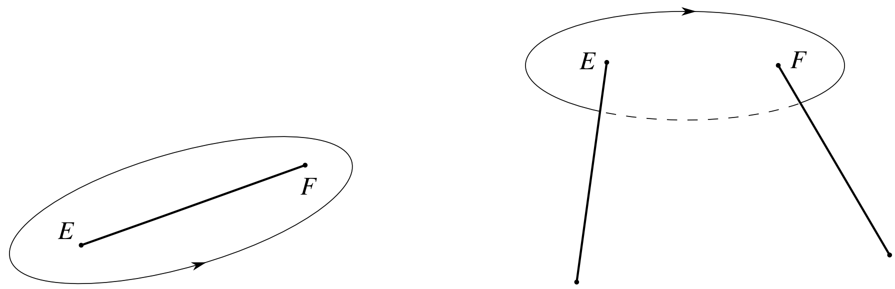
$$c = \frac{x - y}{2}, \quad d = \frac{x + y}{2},$$

where

$$x = \arcsin \frac{2Q - R - S}{S - R}, \quad y = \arcsin \frac{2P - R - S}{S - R},$$

we obtain an integral from -1 to 1 and we can use the Clenshaw-Curtis quadrature. The advantage of the above substitution is that when we integrate between points R and S, i. e. when removing singularities at the ends

[27] In the theory of evolution equations, the situation when pairs of branch points are collapsing and the surface is degenerating, the so-called solitonic limit, is paricularly important.



(a) Connected points (b) Not connected points

Figure 7: Line integration between points connected or not connected by a branch-cut. Solid line represents a path of integration on the +-sheet.

of the line of integration, the constants $c = \pi/2$ and $d = 0$ are real in contrast to the substitution used in [11, 12].

Hence, if we integrate (8.5) between two branch points E and F in the situation when there are 2 other branch points E' near E and F' near F, we divide the integration line in two and we integrate two integrals. In the first integral between $P = E$ and $Q = (E + F)/2$, we set $R = E$ and $S = E'$ and we use the substitution (8.11) and the Clenshaw-Curtis quadrature. In the second integral between $P = (E + F)/2$ and $Q = F$, we set $R = F$ and $S = F'$ and we integrate using the Clenshaw-Curtis quadrature again. The only problem with the substitution (8.11) is that the integration path $t \to a\sin(ct+d)+b$ is not a straight line between P and Q anymore[28] and can in principle approach another critical point of the integrand. Fortunately, the path stays close enough to the original straight line.

To compute the integral over a whole loop, we have to decompose the loop into a sequence of lines between the branch points of the surface. The integral along a line between two branch points connected by a branch-cut is equal to one half of the integral along a loop containing only the two branch points in the counter-clockwise direction on the +-sheet (or in the clockwise direction on the −-sheet), see Figure 7(a). On the other hand, the integral along a line between two branch points E, F not connected by a branch-cut is equal to one half of the integral along a loop containing only the two branch points that goes from the branch-cut corresponding to E to the branch-cut corresponding to F on the +-sheet and back to the branch-cut at E on the −-sheet, Figure 7(b).

In the code, we use the spectral method (8.8) when the line of integration

[28]That's why we cannot use the square root ${}^{(\alpha)}\!\sqrt{\cdot}$ anymore and we have to continue the square root analytically.

between points E and F is far enough from other branch points, because it is the fastest methods. In other cases we use the latter approach with Clenshaw-Curtis quadrature, but it is slightly slower since $g+1$ DCTs of data of length $N+1$ are needed.

Once we have computed the a- and b-periods of the differentials $\nu_1, \ldots, \nu_{g+1}$, we compute the matrix Π of the b-periods of the normalized basis of holomorphic differentials and the b-periods $\int_{b_j} \omega_{\infty^+\infty^-}$ of the Abelian differential of the third kind $\omega_{\infty^+\infty^-}$. Let $\tilde{A}_{jk} = \int_{a_j} \nu_k$ be the matrix of a-periods of the holomorphic differentials $\nu_1, \ldots, \nu_g$. Then the matrix Π of b-periods is given as

$$\pi_{jk} = \int_{b_j} \omega_k = \tilde{A}^{-1}_{lk} \int_{b_j} \nu_l \tag{8.13}$$

and the b-periods of $\omega_{\infty^+\infty^-}$ are given as

$$\int_{b_j} \omega_{\infty^+\infty^-} = \int_{b_j} \nu_{g+1} - \pi_{jk} \int_{a_k} \nu_{g+1}.$$

8.1.3 Theta function

Although there are implementations for the evaluation of theta functions ([6]) in Maple 8, for example, there are no such implementations in Mathematica 5.0[29]. We implemented a simple method for evaluation of theta functions of general genus using the defining series (4.1). In most cases, the number of terms necessary to reach the machine precision is at most 15 in each direction which allows evaluation of theta function up to genus 4 with time that is small compared to the time needed to compute the parameters of the Riemann surface. The number of terms in the series (4.1) is determined by the condition that they are strictly smaller than some threshold value ε which is required to be smaller then the machine precision. The condition for the maximal norm of N necessary in (4.1) to obtain the desired precison reads

$$\|N\| > \frac{\|z\| + \sqrt{\|z\|^2 - \frac{1}{\pi}\lambda_0 \ln \varepsilon}}{\lambda_0},$$

with λ_0 being the smallest[30] eigenvalue of $\operatorname{Im} \Pi$.

8.1.4 Mass and angular momentum

The mass and the angular momentum of the spacetime described by a solution of the Ernst equation can be inferred from the behavior of the Ernst

[29] We are not aware of any of them.

[30] Recall that $\operatorname{Im} \Pi > 0$.

potential $\mathcal{E}$ at infinity, i. e. in the limit $|\xi| \to \infty$. The asymptotic expansion of the Ernst potential $\mathcal{E}$ is given by the formula (8.4). With the substitution $\xi = \zeta - \mathrm{i}\rho = k/t$ with $k \in \mathbb{C}$, $\operatorname{Im} k < 0$, it suffices to analyze the behavior of $\mathcal{E}(\xi(t), \bar{\xi}(t))$ at $t \to 0+$.

Numerically, we sample the function $\mathcal{E}(\xi(t), \bar{\xi}(t))$ at some points $t_l > 0$ close enough to zero and we fit the polynomial $p(t) = \sum_{n=0}^{d} p_n t^n$ of degree $d \geq 3$ to the samples by the least-square method. Then we deduce the constants m (mass) and J (angular momentum) from the coefficients of the fitted polynomial. This procedure yields a very quick and reliable determination of the physical parameters of spacetime.

8.2 Numerical results

The implemented code was used to obtain Ernst potentials on hyperelliptic surfaces of genus 1 and genus 2. The branch points of the surfaces were $\{\xi, \bar{\xi}, \mathrm{i}, -\mathrm{i}\}$ and $\{\xi, \bar{\xi}, \mathrm{i}-1, -\mathrm{i}-1, \mathrm{i}+1, -\mathrm{i}+1\}$. Characteristic p, q was chosen to yield a 'realistic' Ernst potential. Namely, $p = \{0.2\}, q = \{0.\}$ for genus 1 and $p = \{0.2, 0.\}, q = \{0., 0.\}$ for genus 2.

The first step is a computation of the matrix of b-periods Π and the value of $\varphi(\infty^+) = -\varphi(\infty^-)$. For the integration, we constantly used Chebyshev approximation by 32 polynomials. The speed of our code was 70 (resp. 30) different positions of $\xi, \bar{\xi}$ per second for genus 1 (resp. genus 2) solution on a low-end computer. We tested the performance of the numerical routines by introducing two quantities – the norm of the asymmetric part of Π, denoted by b_Δ,

$$b_\Delta = \left\| \frac{\Pi - \Pi^T}{2} \right\|,$$

and the norm of the sum of a-periods of all differentials ν_j, $j = 1, \ldots, g+1$ together with the periods of all differentials around a curve a_0 with $a_0 \cdot b_j = -1$,

$$a_\Delta^2 = \sum_{k=1}^{g+1} \left(\sum_{j=0}^{g} \int_{a_j} \nu_k - 2\pi \mathrm{i} \delta_{k(g+1)} \right)^2 .$$

The term $-2\pi\mathrm{i}\delta_{k(g+1)}$ is there because ν_{g+1} has residue 1 at ∞^+. Both quantities should yield zero. Their magnitude is thus a measure of the error of estimation of Π and $\varphi(\infty^+)$. With a good choice of integration paths and the integration method – spectral method or Clenshaw-Curtis quadrature – we got the value a_Δ as low as the machine precision, i. e. $a_\Delta \sim 10^{-15}$. The magnitude of b_Δ was higher, i. e. $b_\Delta \sim 10^{-8}$. As an example of the dependence of numerical precision on a correct choice of integration paths, we

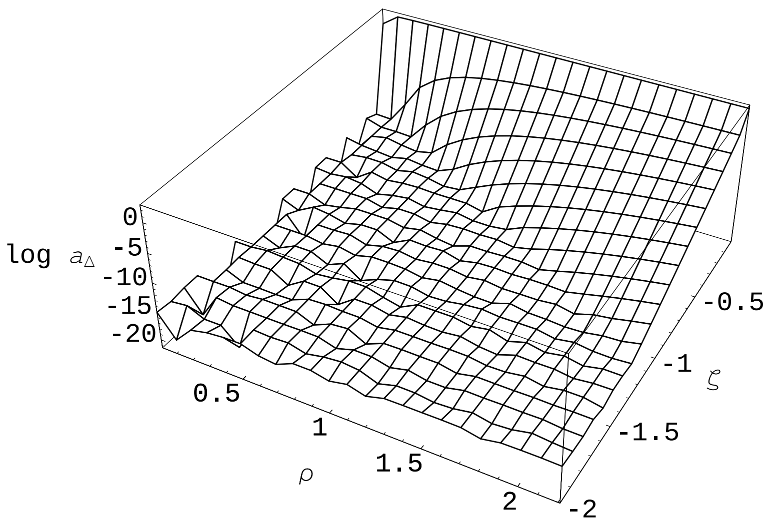


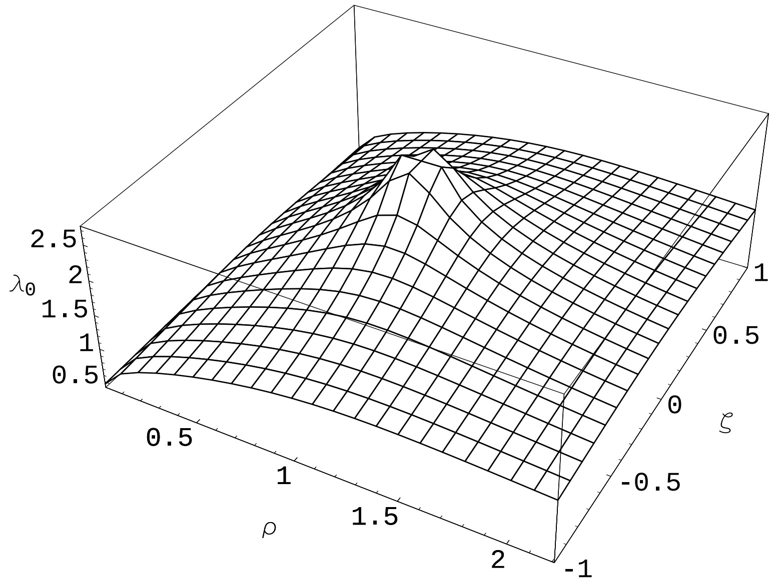
Figure 8: The error of numerical integration measured by a_Δ

present a plot of a_Δ for the hyperelliptic Riemann surface with branch points $\{\xi, \bar{\xi}, -0.00001\mathrm{i}, 0.00001\mathrm{i}\}$ (solitonic limit) versus the value of $\xi(=\zeta - \mathrm{i}\rho)$, see figure 8. It is clear that when some other branch points are too close to the integration path (in this case the branch points $-0.00001\mathrm{i}$ and $0.00001\mathrm{i}$ to the line between ξ and $\bar{\xi}$), the numerical integration fails. The dependence of the minimal eigenvalue λ_0 of $\Im\Pi$ is interesting as well. Figures 9(a) and 9(b) show this dependence for hyperelliptic surfaces of genus 1 and 2, respectively. One can see that λ_0 diverges at the branch points in the plot range.

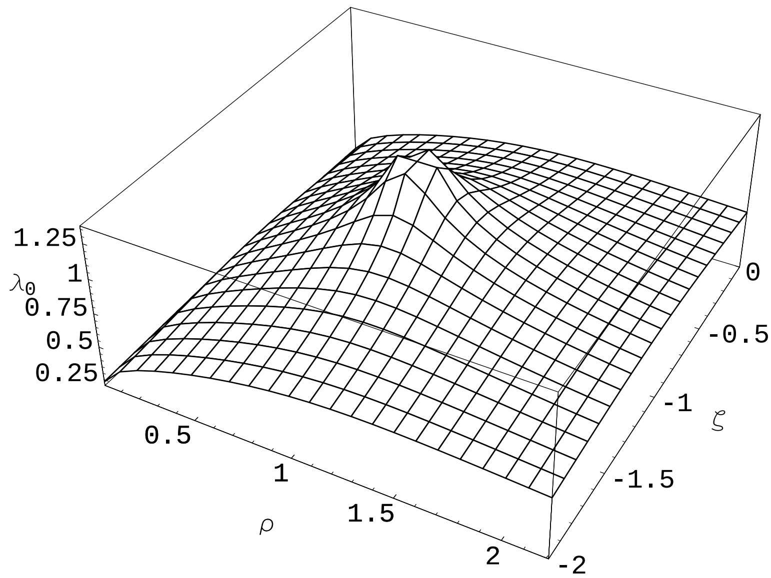
Problems appeared in the evaluation of Ernst potential from the quantities on a surface. Although the obtained potentials have the right qualitative behavior (asymptotic flatness, $\mathcal{E} \to 1$ when $\xi \to \infty$, differentiability everywhere except points where conditions of theorem 7.1 do not hold), they do not obey the Ernst equation exactly. These problems are probably caused by an incorrect implementation of function (8.3) and we have not been able to solve them yet. As an illustration of our results, we present plots of the Ernst potentials for genus 1 and genus 2 solutions on surfaces with branch points mentioned above, see figure 10 and 11.

The genus 1 solution possess reflection symmetry because it obeys [30]

$$\mathcal{E}(\rho, \zeta) = \bar{\mathcal{E}}(\rho, -\zeta).$$



(a) Genus 1

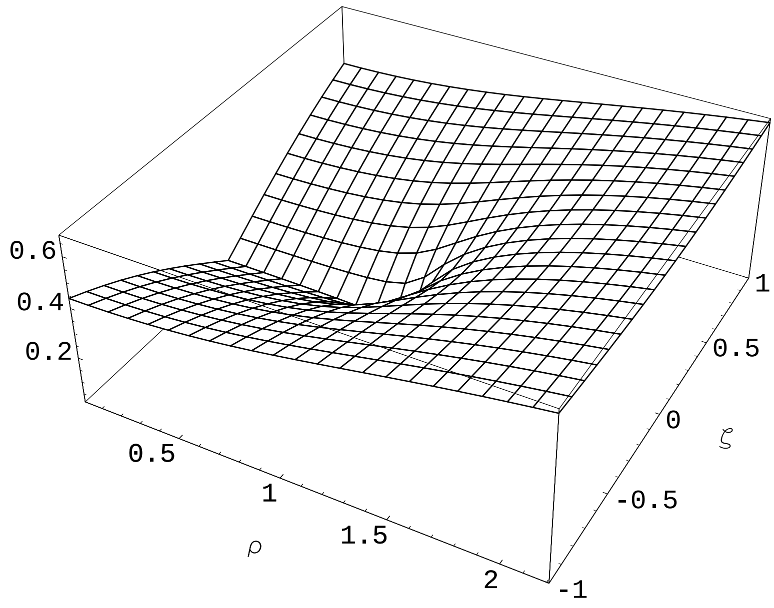


(b) Genus 2

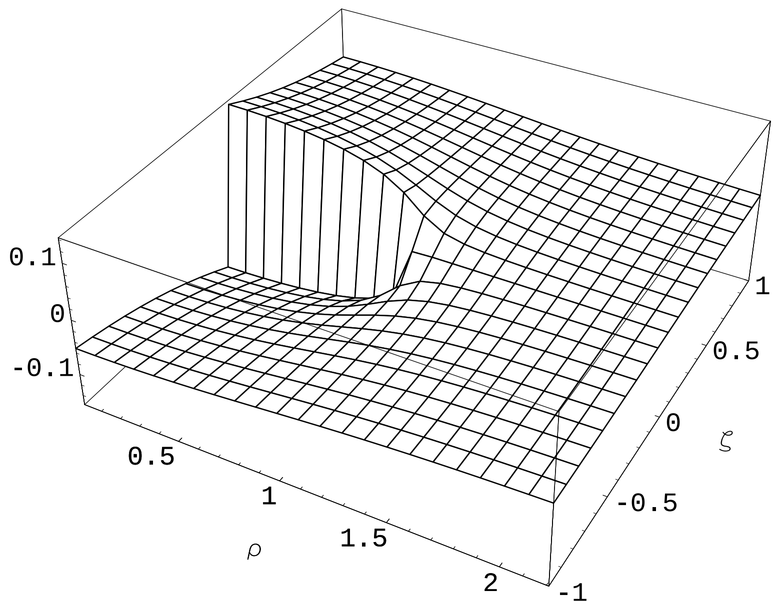
Figure 9: Minimal eigenvalue of the matrix of b-periods Π

Imaginary part of $\mathcal{E}$ has a jump across the line $\zeta = 0$, $\rho \in [0, 1]$, whereas the real part is continuous there. There is a possibility that this jump can be explained by a thin disk of matter in this region. This conjecture is not, however, verified. A study of the asymptotic behavior of $\mathcal{E}$ for $\xi \to \infty$ suggests, that the mass of the object is purely imaginary, therefore very exotic.

The genus 2 solution is presented here only in the range $\zeta < 0$ because the code is not able to change the integration paths automatically yet. In this case, a jump across the line $\zeta \in [-1, 1]$, $\rho = 1$ is present in both real and imaginary parts of $\mathcal{E}$.

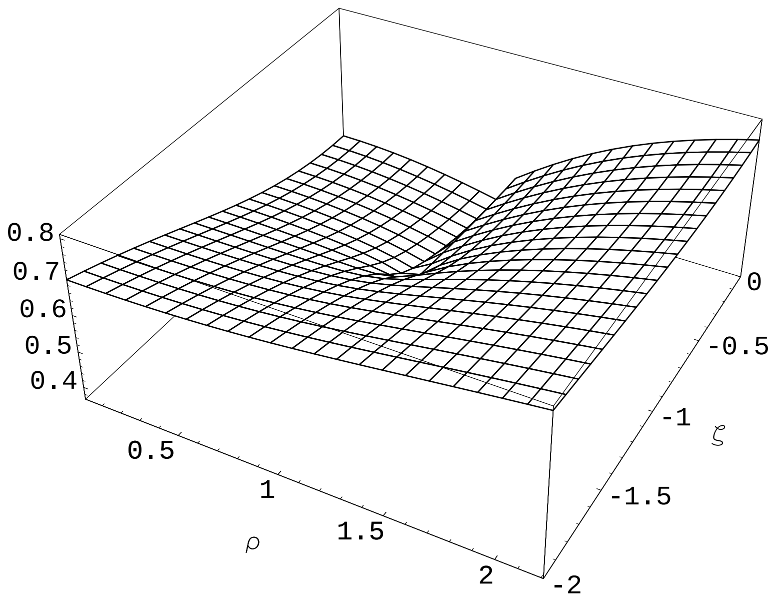


(a) $\Re\mathcal{E}$

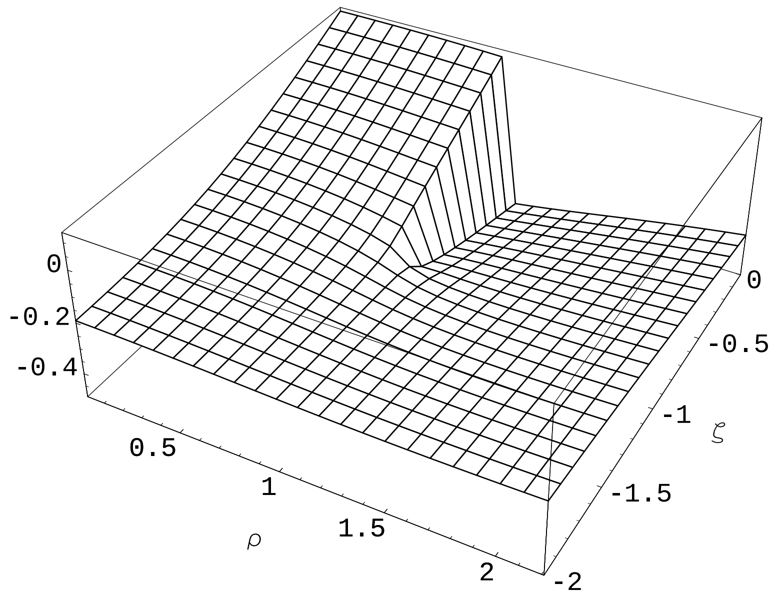


(b) $\Im\mathcal{E}$

Figure 10: The Ernst potential of a genus 1 solution



(a) $\Re\mathcal{E}$



(b) $\Im\mathcal{E}$

Figure 11: The Ernst potential of a genus 2 solution

9 Conclusions

We investigated algebro-geometric methods for construction of solutions to the Ernst equation, in particular using the scalar Riemann-Hilbert problem on hyperelliptic Riemann surfaces, following the approach in [21]. We gave more detailed and perhaps more precise proofs of two key theorems that summarize the analytic properties of matrix functions that satisfy linear systems for the Ernst equation. We also discussed relevance of hyperelliptic Riemann surfaces.

We implemented "fast" and accurate code for evaluation of quantities of interest on hyperelliptic surfaces. Although the code is far from complete, we obtained some results and plots comparable to a similar program by Frauendiener & Klein [11, 12].

There are still many open problems. In this work, we did not address physical properties of the theta solutions although more detailed investigation of solutions of higher genus has not been published yet. There is a high chance to identify subclasses of these solutions describing more complex physical systems, in particular a black hole–disk system. There is no direct way to infer the jump data from the boundary value problem one wants to solve. The explicit form of the theta solutions, however, possibly offers a different approach to boundary value problems: one can try to identify the free parameters in the solutions from the boundary problem.

An interesting problem is also a construction of theta solutions to the Einstein-Maxwell equations. In this case, the linear system is similar to the linear systems for the Ernst equation, but this time it is a system for a 3×3 matrix . The solutions can be constructed using the recent solution to the Riemann-Hilbert problem with quasi-permutation monodromies [25]. The solution lives on a 3-sheeted cover of $\mathbb{CP}^1$ and therefore the structure of this surface will not be as simple as in the case of hyperelliptic surfaces. More involved theory will have to be used and the numerical code will have to be extended to manage numerics on such surfaces.

The code can also be improved considerably so it will be able to handle general hyperelliptic surfaces without any assumptions on reality properties or genus. A black-box program for a general use, working with general hyperelliptic surfaces without any need of instigation of the surface, would be very helpful for astrophysical and other applications.

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