MOTION BY CRYSTALLINE-LIKE MEAN CURVATURE: A SURVEY

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Abstract. We consider a class of anisotropic curvature flows called crystalline curvature flows. We present a survey on this class of flows with special emphasis on the well-posedness of its initial value problem.

Contents

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1. Introduction

The famous mean curvature flow was introduced by W. W. Mullins [136] to model the motion of an antiphase grain boundary in annealing metals. Its governing equation is called the mean curvature equation and it is an equation for a one-parameter family of hypersurfaces $\{\Gamma_t\}$ (an evolving hypersurface) in $\mathbb{R}^n$ which imposes that the normal velocity V equals the mean curvature κ, i.e.,

$$V = \kappa \quad \text{on} \quad \Gamma_t;$$

here, the curvature and the velocity is taken in the direction of the normal vector field ν of Γ_t. This equation can be interpreted as a steepest descent flow of the surface area. In materials science the surface area is considered as an interfacial energy of the grain boundary. It is quite natural to consider anisotropic effects. For this purpose, one considers the anisotropic interfacial energy

$$I(\Gamma) = \int_\Gamma \sigma(\nu)\, d\mathcal{H}^{n-1},$$

where σ is a given positive function called the interfacial energy density; here, $d\mathcal{H}^{n-1}$ is the surface area element of a hypersurface Γ. Its first variation is called the anisotropic mean curvature denoted by κ_σ; this is often called the weighted mean curvature. If one replaces the mean curvature by the anisotropic mean curvature in the mean curvature flow equation, the resulting equation is of the form

$$V = \kappa_\sigma \quad \text{on} \quad \Gamma_t. \tag{1.1}$$

In general, this equation may not be parabolic even if σ is smooth. We consider the one-homogeneous extension of σ in $\mathbb{R}^n$ and still denote it by σ, i.e.,

$$\sigma(p) = |p|\sigma(p/|p|), \quad p \in \mathbb{R}^n \setminus \{0\}. \tag{1.2}$$

If σ is convex, the equation (1.1) is at least degenerate parabolic. Although the problem when σ is not convex is interesting, we do not touch this problem in this paper. The reader is referred to [22] for such an ill-posed problem.

The anisotropic mean curvature flow can be considered as the mean curvature flow in a Minkowski metric or a Finsler metric. In this case, V should be replaced by the Minkowski normal velocity. If one uses the Euclidean

normal velocity, it is of the form

$$V = \sigma\kappa_\sigma;$$

see [30] for this perspective.

The curvature flow is not restricted to the form (1.1). For second-order models, a general form of the flow is

$$V = g(\nu, \kappa_\sigma) \tag{1.3}$$

with g non-decreasing in the second variable. A typical example in thermodynamics is

$$V = M(\nu)(\kappa_\sigma + C)$$

with mobility $M(\nu) > 0$ and a driving force C, where C is a constant [99], [10]. There are several other examples when g is nonlinear in κ_σ. For example,

$$V = |\kappa_\sigma|^{\alpha-1}\kappa_\sigma$$

with some positive number α. We shall discuss these examples in Section 2.

For later convenience, we say that $\sigma : \mathbb{R}^n \to [0, \infty)$ is an *anisotropy* if σ is positively one-homogeneous, convex and $\sigma > 0$ outside the origin. Note that we do not assume that σ is even. By definition, σ satisfies (1.2) and the *Frank diagram*

$$F_\sigma = \{p \in \mathbb{R}^n \mid \sigma(p) \le 1\}$$

is bounded, convex and contains the origin as an interior point.

For many applications, especially in low temperature physics, it is often considered the case that σ is not C^1. An extreme case is that the anisotropy σ is (purely) *crystalline*, i.e., σ is piecewise linear so that F_σ is a convex polytope. A crystalline mean curvature flow is formally (1.3) when anisotropy σ is crystalline. In mathematical community, it was introduced by J. E. Taylor [156] and independently by S. B. Angenent and M. E. Gurtin [10] around 1990.

One might be curious on the value of κ_σ when σ is crystalline. To motivate it we consider an anisotropic isoperimetric problem of the form
"Find a shape D in $\mathbb{R}^n$ with fixed volume which minimizes the surface energy $I(\Gamma)$ with $\Gamma = \partial D$."
This problem was first studied by Wulff [167] and it turns out that the minimizer is the *Wulff shape*

$$W_\sigma = \bigcap_{|m|=1} \{x \in \mathbb{R}^n \mid x \cdot m \le \sigma(m)\},$$

which is the polar of F_σ. This has been proved in quite general setting; see e.g. [155], [62]. For recent developments related to optimal transport theory,

see [61]. Note that if σ is crystalline so that F_σ is a polytope, then W_σ is also a polytope. For a smooth anisotropy, one observes that the anisotropic κ_σ on the surface of W_σ is a non-zero constant, and so W_σ plays the same role as a ball for the usual curvature. More precisely, if one takes ν inward, $\kappa_\sigma = n-1$. If σ is crystalline, then W_σ is a polytope. Nevertheless, κ_σ should not be zero. This simple observation shows that the value κ_σ cannot be determined by infinitesimal quantities like tangent and second fundamental form of the surface. We say that (1.3) is a *crystalline (mean) curvature flow (equation)* if σ is crystalline.

We now consider a simple formal example of a crystalline curvature flow for a graph-like curve. For later convenience, we write the equation (1.1) when Γ_t is given as the graph of a function $w = w(x', t)$, i.e., $x_n = w(x', t)$ for $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$. The upward normal velocity is given as

$$V = \frac{w_t}{(1 + |\nabla' w|^2)^{1/2}},$$

where $w_t = \partial w/\partial t$, $\nabla' w = (\partial_{x_1} w, \dots, \partial_{x_{n-1}} w)$, $\partial_{x_j} = \partial/\partial x_j$, $w_{x_j} = \partial_{x_j} w$. The anisotropic mean curvature is formally of the form

$$\kappa_\sigma = -\operatorname{div}_{\Gamma_t} \zeta(\nu) \quad \text{with} \quad \zeta(\nu) = (\nabla_p \sigma)(\nu),$$

where $\nabla_p \sigma$ denotes the gradient of σ, i.e., $\nabla_p \sigma = (\partial_{p_1} \sigma, \dots, \partial_{p_n} \sigma)$ for anisotropy $\sigma = \sigma(p_1, \dots, p_n)$. The divergence $\operatorname{div}_{\Gamma_t}$ denotes the surface divergence, i.e.,

$$\operatorname{div}_{\Gamma_t} X = \operatorname{trace}\big((I - \nu \otimes \nu)\nabla X\big);$$

here, we extend X in a tubular neighborhood of Γ_t in a suitable way and ∇X denotes its Jacobi matrix. This value is independent of the extension; see e.g. [83]. In our setting,

$$\operatorname{div}_{\Gamma_t} \zeta(\nu) = \sum_{\ell=1}^{n-1} \frac{\partial}{\partial x_\ell} \left(\frac{\partial \sigma}{\partial p_\ell}(\nu) \right)$$

where $\nu = (-\nabla' w, 1)/\left(1 + |\nabla' w|^2\right)^{1/2}$. Indeed,

$$\operatorname{trace}(\nu \otimes \nu \nabla \zeta) = \sum_{i,j=1}^{n} \nu_i \nu_j \frac{\partial}{\partial x_i} \left((\partial_{p_j} \sigma)(\nu) \right) = \sum_{i,j,\ell=1}^{n} \nu_i \nu_j (\partial_{p_j} \partial_{p_\ell} \sigma)(\nu) \partial_{x_j} \nu^\ell = 0$$

since $\sum_{j=1}^n \nu_j \partial_{p_j} \left((\partial_{p_\ell} \sigma)(\nu) \right) = 0$ by positively zero-homogeneity[1] of $\partial_{p_\ell} \sigma$. Moreover, since $\partial_{p_\ell} \sigma(\nu)$ is independent of x_n, we have the desired identity.

[1] Let s be a real number. A function f allowing values $\pm\infty$ defined in a vector space V is called *positively s-homogeneous* if $f(\lambda v) = \lambda^s f(v)$ holds for all $\lambda > 0$ and $v \in V$. The function f is allowed to take the value $+\infty$ especially when V is an infinite dimensional space. Indeed, in Section 5.1, we consider a total variation type energy in L^2 space, which is an example of positively one-homogeneous but take the value $+\infty$ somewhere.

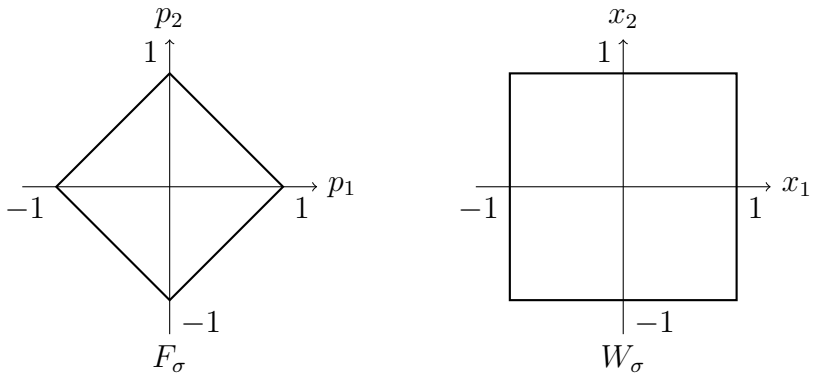

FIGURE 1. The Frank diagram and the Wulff shape for $\sigma(p) = |p_1| + |p_2|$.

If Γ_t is a graph-curve in $\mathbb{R}^2$, then

$$-\operatorname{div}_{\Gamma_t} \zeta = -\partial_{x_1}\left(\frac{\partial\sigma}{\partial p_1}(-w_x, 1)\right),$$

since $\nabla_p\sigma$ is positively zero-homogeneous.

We now observe that (1.1) is formally of the form

$$\frac{w_t}{(1+w_{x_1}^2)^{1/2}} = -\partial_{x_1}\left(\frac{\partial\sigma}{\partial p_1}(-w_{x_1}, 1)\right). \tag{1.4}$$

If $\sigma(p) = |p|$, then

$$\frac{\partial\sigma}{\partial p_1}(p) = \frac{p_1}{|p|} \quad \text{so that} \quad \frac{\partial\sigma}{\partial p_1}(-w_{x_1}, 1) = -\frac{w_{x_1}}{(1+w_{x_1}^2)^{1/2}},$$

which yields a curve-shortening equation for a graph-like curve $\Gamma_t : x_2 = w(x_1, t)$, i.e.,

$$\frac{w_t}{(1+w_{x_1}^2)^{1/2}} = \partial_{x_1}\left(\frac{w_{x_1}}{(1+w_{x_1}^2)^{1/2}}\right) \quad \text{or} \quad w_t = \frac{w_{x_1x_1}}{1+w_{x_1}^2}.$$

We are interested in the case when σ is crystalline. Let us consider

$$\sigma(p) = |p_1| + |p_2|$$

so that the Frank diagram F_σ is a square whose vertices are $(\pm 1, 0)$ and $(0, \pm 1)$; see Figure 1 for F_σ and the corresponding Wulff shape W_σ. The derivative $\frac{\partial\sigma}{\partial p_1}$ must be interpreted in a suitable generalized sense, for example, as the subdifferential of σ; see (1.6) below. Then (1.4) becomes $w_t = (1+w_{x_1}^2)^{1/2}\partial_{x_1}(\operatorname{sgn} w_{x_1})$. Since it is expected that $\partial_{x_1}(\operatorname{sgn} w_{x_1}) = 0$ away from the set $\{w_{x_1} = 0\}$, it is formally equivalent to

$$w_t = \partial_{x_1}(\operatorname{sgn} w_{x_1}), \tag{1.5}$$

where $\operatorname{sgn} p_1 = p_1/|p_1|$. This equation is a total variation flow equation in one-dimensional setting. If one calculates the right-hand side formally, then

(1.5) is

$$w_t = 2\delta(w_{x_1})w_{x_1x_1},$$

where δ denotes Dirac's delta. This shows

$$\partial_{x_1}(\operatorname{sgn} w_{x_1}) = (1 + w_{x_1}^2)^{1/2}\partial_{x_1}(\operatorname{sgn} w_{x_1}).$$

However, the quantity $\delta(w_{x_1})$ is undefined because it is a pull-back of the delta measure although it suggests the diffusion coefficient equals zero if w_{x_1} is not equal to zero. In other words, the place where w_{x_1} is not zero does not move. To see the speed where w_{x_1} is zero, let us consider a special (Lipschitz) profile $x_2 = w_0(x_1)$ which takes the minimum value on $[a, b]$ and $w_{0x_1} > 0$ (resp. $w_{0x_1} < 0$) in $x_1 > b$ (resp. $x_1 < a$), where $a < b$ (Figure 2). We try to move this function by (1.5). Since it is natural to assume that

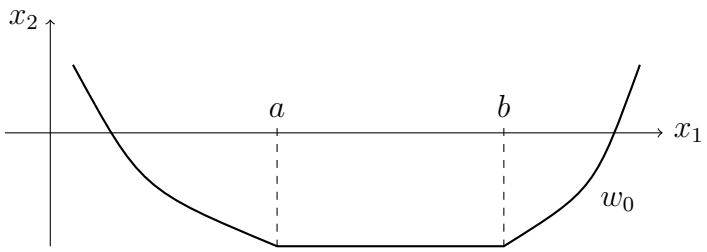

FIGURE 2. The graph of w_0.

the speed equals zero outside $[a, b]$, the important thing is to calculate the speed on $[a, b]$. Here we put the ansatz:
"The speed w_t on $[a, b]$ is spatially constant."
In other words, a flat part (called facet) stays as a facet and no bending nor facet splitting occurs.

We integrate (1.5) in a neighborhood of $[a, b]$, i.e., $(a - \varepsilon, b + \varepsilon)$ with small $\varepsilon > 0$ and obtain at $t = 0$

$$\int_{a+\varepsilon}^{b+\varepsilon} w_t dx = \int_{a+\varepsilon}^{b+\varepsilon} \partial_{x_1}(\operatorname{sgn} w_{0x_1})dx = \operatorname{sgn} w_{0x_1}(b + \varepsilon) - \operatorname{sgn} w_{0x_1}(a - \varepsilon)$$
$$= 1 - (-1) = 2.$$

By our ansatz, the left-hand side is of the form

$$w_t(b - a)$$

as $\varepsilon \to 0$. Thus, we obtain

$$w_t = 2/(b - a).$$

The right-hand side is a nonlocal quantity and this is a one-dimensional version of the *Cheeger ratio* $\mathcal{H}^{n-1}(\partial\Omega)/\mathcal{L}^n(\Omega)$ defined for a domain Ω in $\mathbb{R}^n$, where $\mathcal{L}^n(\Omega)$ denotes the Lebesgue measure of Ω while $\mathcal{H}^{n-1}(\partial\Omega)$ denotes

the $(n-1)$-dimensional Hausdorff measure of the boundary $\partial\Omega$ of Ω. We now observe that the crystalline curvature should be determined by a semilocal quantity like Cheeger ratio if one assumes the ansatz.

In a one-dimensional setting, this ansatz is justified in the sense that such a profile is approximated by a solution of a uniformly parabolic equation which approximates the equation (1.5). For example, order-preserving property called comparison principle is expected to hold. However, in a higher dimensional setting, as we see later, this ansatz is no longer appropriate. For example, this ansatz violates the comparison principle.

For curve evolutions, using this ansatz J. E. Taylor [156] and independently S. B. Angenent and M. E. Gurtin [10] introduced a special class of polygonal curves called admissible. We say that an oriented polygon is *admissible* if the orientation (normal ν) of each facet (edge) is one of that in ∂W_σ and the orientation of adjacent facets should be adjacent in ∂W_σ. Here W_σ is the Wulff shape associated with anisotropy σ and it is a convex polygon if σ is crystalline. If the second condition (called adjacency condition) is not required, one expects that new facets may be created because of a strong curvature effect. We shall discuss this point in Section 3. Let $\{\Gamma_t\}$ be a smooth family of admissible polygons. In other words, vertices of Γ_t are assumed to move C^1 in time t. The motion of vertices is completely determined by the crystalline flow equation (1.3). Here, κ_σ of each facet with normal ν is assumed to be equal to $\chi\Delta/L$, where L is the length of the facet and Δ is the length of the facet of W_σ with normal ν; χ takes values in $\{-1, 0, +1\}$ depending upon convexity near the facet. Since L depends upon vertices, combining these equations, a system of ordinary differential equations (ODEs) for vertices or lengths is obtained. Its initial value problem is uniquely solvable at least when g is (locally) Lipschitz continuous. For later convenience, we say that $\{\Gamma_t\}$ is a crystalline flow if Γ_t is a smooth family of admissible polygons satisfying the system of these ODEs. However, there is a chance that in finite time a facet disappears. Fortunately, in many cases at the time when a facet disappears, Γ_t is still admissible so one is able to continue to solve the system of ODEs with fewer facets. This approach is simple and it is easy to compute the crystalline flow [156], [160], [157]. Moreover, it satisfies the desired property like comparison principle which says that if one admissible polygon encloses another, then the corresponding crystalline flow starting from these polygons keeps this order; see [160], [85].

There is another approach based on the theory of maximal monotone operators initiated by Y. Kōmura [123] and developed by H. Brezis [31] and others in late 1960s and 1970s. A basic theory asserts the unique global-in-time solvability of the initial value problem for the gradient flow equation whose "energy" $\mathcal{E}$ is a convex, lower semicontinuous functional in a Hilbert space H equipped with an inner product $\langle\cdot,\cdot\rangle$ so that $\|f\|_H^2 = \langle f, f\rangle$. More precisely, it is a solvability for the system $w_t \in -\partial\mathcal{E}(w)$ where $\partial\mathcal{E}(w)$ is the subdifferential of $\mathcal{E}$ at w, which is an extended notion of a differential of $\mathcal{E}$. It is defined as

$$\partial\mathcal{E}(w) = \{f \in H \mid \mathcal{E}(w+h) - \mathcal{E}(w) \geq \langle f, h\rangle \text{ for all } h \in H\}. \tag{1.6}$$

Note that $\mathcal{E}$ may not be differentiable so that $\partial\mathcal{E}(w)$ may not be a singleton. However, the solution is unique and it "knows" how to grow even though the evolution law looks ambiguous. Actually, the solution is right differentiable in time and its speed equals to the minimal section (canonical restriction) $\partial^0\mathcal{E}(w)$ of $\partial\mathcal{E}(w)$, i.e.,

$$\partial^0\mathcal{E}(w) = \operatorname{argmin}\bigl\{\|f\|_H \;\big|\; f \in \partial\mathcal{E}(w)\bigr\},$$

which is uniquely determined. In [63], it is shown that if $\{\Gamma_t\}$ is given as the graph of a periodic function of one variable, then the equation $V = M(\nu)\kappa_\sigma$ can be written as a gradient flow system. Moreover, the speed given by the general theory is the same as the one given in the ansatz on a facet. This suggests the approach by [156], [10] is quite natural. In fact, it is shown in [63] that the crystalline flow is obtained as a limit of approximate solutions solving a usual uniformly parabolic problem approximating the original problem. This justifies the ansatz for curve evolution. The proof is based on a general convergence theory for gradient flow systems developed by [33] and [165]. To apply the theory, it suffices to prove that the approximating energy $\mathcal{E}^\varepsilon$ converges to $\mathcal{E}$ in the sense of Mosco, i.e., it satisfies

(i) sequential lower semicontinuity under weak topology:

$$\mathcal{E}(w) \leq \varliminf_{\varepsilon\downarrow 0} \mathcal{E}^\varepsilon(w_\varepsilon) \quad \text{for} \quad w_\varepsilon \rightharpoonup w \ (\text{as } \varepsilon \to 0); \tag{1.7a}$$

(ii) existence of a strong recovery sequence: for any $v \in H$, there is $v_\varepsilon \to v$ as $\varepsilon \to 0$ such that

$$\mathcal{E}(v) = \lim_{\varepsilon\downarrow 0} \mathcal{E}^\varepsilon(v_\varepsilon). \tag{1.7b}$$

The nonlocal property of the speed related to a total-variation-type singular energy was also observed in [102].

If the flow equation is written as a gradient flow of a convex, lower semicontinuous functional in a Hilbert space, one is able to calculate the speed by calculating the minimal section. It is a kind of obstacle problem as we will see later. Reflecting this idea, G. Bellettini, M. Novaga and M. Paolini [25] gave an example that the speed of a facet may not be a constant on a facet. In other words, the quantity κ_σ may not be a constant on a facet since otherwise it would contradict a comparison principle. Later, they gave a characterization of non-constancy of κ_σ on a facet depending on shape. To illustrate the problem, let us consider a closely related problem: the total variation flow equation

$$w_t = \operatorname{div}(\nabla w/|\nabla w|) \tag{1.8}$$

on an n-dimensional torus $\mathbb{T}^n = \Pi_{i=1}^n(\mathbb{R}/\omega_i\mathbb{Z})$, $\omega_i > 0$ $(i = 1, \dots, n)$. Except Section 9, we shall assume $\omega_i = 1$ for simplicity. It can be interpreted as a gradient flow of the total variation energy

$$E[w] = \int_{\mathbb{T}^n} |\nabla w| := \sup\left\{ \int_{\mathbb{T}^n} w \operatorname{div} z \, dx \;\middle|\; |z(x)| \le 1,\ z \in C^1(\mathbb{T}^n, \mathbb{R}^n) \right\}$$

for an L^2 function w. We set the energy $\mathcal{E}$ in the Hilbert space $H = L^2(\mathbb{T}^n)$ such that $\mathcal{E} = E$. Then, it is not difficult to see that $\mathcal{E}$ is convex and lower semicontinuous in $H = L^2(\mathbb{T}^n)$. The problem (1.8) should be interpreted as

$$w_t \in -\partial\mathcal{E}(w)$$

and there is a unique solution starting from $w_0 \in H = L^2(\mathbb{T}^n)$. The speed is given as the minimal section and we are interested in the value. We restrict ourselves to a facet where w is "convex" in its neighborhood. We fix $t > 0$ and let w take its minimum on a facet, i.e.,

$$F = \left\{ x \in \mathbb{T}^n \;\middle|\; w(x,t) = \min_{y \in \mathbb{T}^n} w(y,t) \right\}.$$

Assume that the boundary of F is smooth. Then it turns out that

$$-\partial^0\mathcal{E}(w)\big|_F = \operatorname{div} z,$$
$$z = \operatorname{argmin}\left\{ \int_F |\operatorname{div}\zeta|^2 \, dx \;\middle|\; \zeta \cdot \nu_F = 1 \text{ on } \partial F,\ |\zeta| \le 1 \text{ in } F \right\}.$$

Here ν_F is the exterior unit normal of F. This is a convex minimization problem but it is of obstacle type because of the constraint $|\zeta| \le 1$. Although the minimizer is not unique, $\operatorname{div} z$ is uniquely determined. The characterization of the minimal section is nontrivial but it can be done for the total variation flow equation. For a detailed explanation, the reader is referred to a very nice book by F. Andreu-Vaillo, V. Caselles and J. M. Mazón [7]. If $\operatorname{div} z$ is

constant, we say that F is *calibrable*. There are several necessary and sufficient conditions; see e.g. [26] for the curvature flow. We shall discuss this topic in Section 5. If it is calibrable, then $\operatorname{div} z$ must be the Cheeger ratio, i.e., $\operatorname{div} z = \mathcal{H}^{n-1}(\partial F)/\mathcal{L}^n(F)$. Indeed, integration by parts yields

$$(\operatorname{div} z)\mathcal{L}^n(F) = \int_F \operatorname{div} z \, dx = \int_{\partial F} z \cdot \nu_F \, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial F).$$

In general, $\operatorname{div} z \in L^\infty \cap BV$ but may be discontinuous as shown in [27], [28]. Since there may exist non-calibrable facets, it took a long time to construct a solution in a general setting. G. Bellettini and M. Novaga [24] introduced a notion of a solution based on distance function reflecting the variational structure and proved its uniqueness. However, its existence is only proved for convex initial data [18]. It is quite recent that the well-posedness problem is settled by two groups through the level-set method, which is the main topic of this survey.

Although there are several approaches to solve the problems by now, they are roughly classified into three main ones. The first approach is to consider a special class of evolving polygons by reducing the problem to a system of ODEs we discussed before. This approach is valid only for curve evolution. The second approach is a variational approach. A simple way is to apply the theory of maximal monotone operators which is restricted for the graph case but it has an advantage to apply to a higher order crystalline flow, for example crystalline surface diffusion equation for a graph-like surface. The reader is referred to [74] for this topic as well as Section 9.2. A variant of this variational approach involving a distance function yields a global well-posedness for convex sets as mentioned before [18].

The third approach is a viscosity approach. This is based on the theory of viscosity solutions, which was originally introduced to characterize the value function of a control problem as a solution of a Hamilton-Jacobi equations; see [47]. The notion of a viscosity solution is based on a comparison principle for the second-order elliptic or parabolic equations which can be degenerate. It does not depend on a variational structure. However, since the crystalline flow is non-local, one needs to adjust the theory. This is not trivial even for an evolution of a curve. In the case of graph-like curves, i.e., Γ_t is given as a graph $w = w(x_1, t)$, the notion of a viscosity solution was adjusted for general crystalline flow when w is periodic in x_1 [68], [70]. It can be approximated by a smoother problem as proved in [71].

This viscosity approach was later extended to a closed curve by adjusting the level-set method [73], [72]. The original level-set method based on

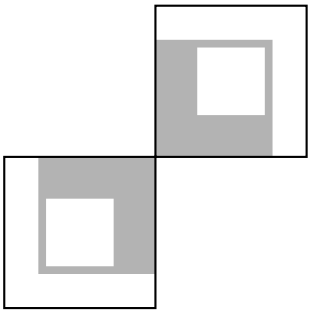

FIGURE 3. Example of a fattening in the crystalline flow for $\sigma(p) = |p_1| + |p_2|$ with initial curve Γ_0 given by the figure-8-shaped solid line. The set Γ_t immediately ($t > 0$) fattens. The gray area denotes Γ_t at $t = 0.8t^*$ where t^* is the extinction time of the individual squares.

the theory of viscosity solution was introduced by [60], [45] for the mean curvature flow equations. The idea of the original level-set method for the mean curvature flow $V = \kappa$ is to consider its level set flow equation

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$$

which requires that *each* level set moves by $V = \kappa$. For a given initial hypersurface Γ_0, one constructs a continuous function u_0 such that Γ_0 is the zero level set of u_0 and solves the level-set flow equation globally-in-time and sets Γ_t as the zero level set of the solution. A unique solvability is guaranteed by the theory of viscosity solutions. Moreover, Γ_t is uniquely determined by Γ_0. However, as already pointed out in [60], Γ_t may have interior even if Γ_0 has no interior, Figure 3; see also [83]. This phenomenon is called fattening and from the point of an evolution of hypersurfaces this is considered a non-uniqueness phenomenon. A basic merit of this approach is to handle a topological change. The generalized solution Γ_t of course agrees with a smooth solution if the latter exists though the proof is less trivial [60], [84]. For a general theory of the level-set method for smooth anisotropy, see [45] or a book [83]. The level-set method itself was introduced by [145] for numerical study and independently by [143] to explain a scaling law of $V = \kappa$. For the development of the numerical approach, see [150] and [144].

It took quite a long time to extend this theory to evolution of a hypersurface mainly because the crystalline curvature κ_σ may not be a constant on a facet. A first breakthrough is done by [81], where the viscosity theory was extended to a total-variation-flow-like equation; see also [80]. Later it was extended to level-set flow equations, [91], [92], and to the case when there is a spatially inhomogeneous driving force term [93]. In the meanwhile, another

approach to construct a level-set flow based on distance functions which goes back to [151] was developed independently. In fact, A. Chambolle, M. Morini and M. Ponsiglione [42] constructed a level-set flow for $V = \sigma\kappa_\sigma$ for very general σ containing crystalline σ as a special case. With M. Novaga they even extended their approach in [40] for more general equations with mobility and spatially inhomogeneous driving force term. In both theories, the theory of maximal monotone operators is reflected in some sense. In the purely viscosity approach by [91], [92], [93], the value κ_σ is defined as the minimal section of the crystalline interfacial energy. In the approach by [42], [40], the distance function from the zero level-set of a solution is interpreted as a supersolution of the original gradient flow of the form $u_t \in -\partial E(u)$, where E is an anisotropic total variation energy with density σ.

We warn the reader that the value κ_σ is not determined completely by the facet F if the problem is spatially inhomogeneous as pointed out by [93]. If there is a non-constant driving force $C = C(x)$, then $\kappa_\sigma + C$ is not just the sum of the two quantities. See [93] for more details and futher references.

We do not intend to cover all topics related to well-posedness for a crystalline flow. Several interesting topics like a crystalline multi-phase curvature flow are missing in this paper. For a multi-phase crystalline flow, see [21].

This paper is organized as follows. In Section 2, we give several model equations for curvature flow equations involving a crystalline curvature. In Section 3, an evolution of a polygon is discussed. In Section 4, some explicit solutions such as self-similar solutions are discussed. In Section 5, we give an approach by the theory of maximal monotone operators. In Section 6, we give an approach based on viscosity solutions. In Section 7, we give an approach based on distance functions. In Section 8, some numerics are given. In Section 9, examples of a fourth-order problem and a volume-preserving flow are discussed.

2. Some models

We begin with second-order models in materials sciences. There is an axiomatic derivation of evolution laws of phase-interfaces involving bulk energy and surface energy with constitutive relation compatible with thermodynamical laws in [10], [99]. Its explicit form is

$$b(\nu, V)V = \kappa_\sigma - f \quad \text{with} \quad b(\nu, V) \geq 0$$

where f is a driving force term coming from bulk interface difference which is assumed to be a constant in [10], [99]. The function b is called a kinetic

coefficient. If $b(\nu, V)$ is independent of V and positive, then it is reduced to

$$V = M(\nu)(\kappa_\sigma + C)$$

with $C = -f$, $M(\nu) = b(\nu)^{-1}$. If $b(\nu, V)$ is taken so that

$$b(\nu, V)V = \log(1 + V)$$

with $f = 0$, this is nothing but the model of thermal grooving of a surface due to evaporation-condensation proposed by W. W. Mullins [137]. Here is a way of derivation. The Gibbs-Thomson law reads

$$\log(p/p_0) = \beta(-\kappa_\sigma)$$

with a positive constant $\beta > 0$. Here p is the pressure and p_0 is the atmospheric pressure. The evolution law is

$$V = M(\nu)(p_0 - p).$$

If $M(\nu) \equiv 1$, $p_0 = 1$, then one gets

$$V = 1 - \exp(-\beta\kappa_\sigma). \tag{2.1}$$

If the right-hand side is linearized around $\kappa_\sigma = 0$, we get $V = \beta\kappa_\sigma$. See the discussion by N. Hamamuki [101]. As we will see later in this section, a model similar to $V = \kappa_\sigma$ was introduced by H. Spohn [152] when σ is a kind of crystalline anisotropy to model evaporation-condensation below the roughening temperature.

Another source of equations stems from image processing. An axiomatic derivation is provided by [4]. For curve evolution, equation

$$V = |\kappa_\sigma|^{\alpha-1}, \kappa_\sigma \quad \alpha > 0 \tag{2.2}$$

is important especially with $\alpha = 1/3$, where the evolution law is invariant under affine transform (not only under rotation, dilation and translation) when σ is isotropic. In higher dimensional case, the corresponding equation should be $V = K^{1/(n+1)}$ where K is the Gauss curvature not the mean curvature. A crystalline Gaussian curvature flow $V = K_\sigma$ has been studied to approximate the Gaussian curvature flow; see e.g. [163]. However, we do not touch this topic in this paper. There are many examples of curvature flows (see e.g. [83, Chapter 1]). In the case that the mean curvature is involved like the inverse mean curvature flow equation, it is easy to generalize

$$V = -1/\kappa_\sigma.$$

If σ is isotropic, then the equation was used to prove the positive mass conjecture [105] since the Geroch mass is monotone under this flow.

We note that the total variation flow

$$w_t = \operatorname{div}'\left(\nabla' w/|\nabla' w|\right)$$

can be understood as a particular case of $V = M(\nu)\kappa_\sigma$ as discussed in the introduction for evolution of graph-like curves. If an evolving surface Γ_t is given as the graph of $w = w(x', t)$, $x' \in \mathbb{R}^{n-1}$, the total variation flow for w can be written as

$$V = M(\nu)\kappa_\sigma$$

with

$$\sigma(p) = |p'| + |p_n| \quad \text{with} \quad p = (p', p_n),$$
$$M(\nu) = \nu_n \quad \text{with} \quad \nu = (\nu', \nu_n).$$

Here $\nu' = -\nabla' w/\left(1 + |\nabla' w|^2\right)^{1/2}$ and $\nu_n = 1/\left(1 + |\nabla' w|^2\right)^{1/2}$.

The model proposed by H. Spohn [152] is almost the same. Here w denotes the height of the crystal surface at x' and at time t. It is of the form

$$w_t = \operatorname{div}'\left(\nabla' w/|\nabla' w|\right) + \beta \operatorname{div}'\left(|\nabla' w|\nabla' w\right),$$

where $\beta > 0$ is a constant. If one writes it in the form of a surface evolution, it is

$$V = M(\nu)\kappa_\sigma$$

with $\sigma(p) = |p'| + \beta|p'|^3/\left(3|p_n|^2\right) + |p_n|$.

There are several fourth-order models. For relaxation of crystal surface, a fourth-order total variation type equation is proposed by [152]. Its explicit form is

$$w_t = -\Delta'\left(\operatorname{div}\left(\nabla' w/|\nabla' w|\right) + \beta \operatorname{div}\left(|\nabla' w|\nabla' w\right)\right),$$

where Δ' denotes the Laplacian in x' variable, i.e., $\Delta' = \operatorname{div}' \operatorname{grad}' = \nabla' \cdot \nabla'$. This equation is derived as a continuum limit of models describing motion of steps on crystal surface as discussed in [142], where a numerical simulation is given. This model describing step-motion is microscopic in the direction of height but macroscopic in the horizontal direction. We refer the reader to a nice review article by R. V. Kohn [120] on this issue. Of course, if $\beta = 0$, this is nothing but the fourth-order total variation flow. This is popular for image processing. For example, Osher-Solé-Vese [146] model gives the fourth-order total variation flow of the form

$$u_t = -\Delta \operatorname{div}\left(\nabla u/|\nabla u|\right) + \lambda(f - u)$$

for $\lambda > 0$, and given f. See also [56] for such a flow, where the well-posedness of the equation is proved by using the Galerkin method. For relaxation phenomena, W. W. Mullins [137] introduced a surface diffusion flow equation;

see also [35] for derivation. It is of the form

$$V = -\operatorname{div}_{\Gamma_t} j, \quad j = -\operatorname{grad}_{\Gamma_t} \rho$$

$$\log(\rho/\rho_0) = \frac{k\mu}{T}, \quad \mu = \kappa_\sigma,$$

where T is a given temperature and ρ_0 is an equilibrium density; k is a positive constant. The quantity j is the mass flux and μ is the chemical potential. The resulting equation is

$$V = \Delta_{\Gamma_t} \exp(-k\kappa_\sigma/T), \quad \Delta_\Gamma = \operatorname{div}_\Gamma \operatorname{grad}_\Gamma; \tag{2.3}$$

here, Δ_Γ denotes the Laplace-Beltrami operator on the surface Γ. We shall set $k = 1$, $T = 1$ for simplicity of presentation to get

$$V = \Delta_{\Gamma_t} \exp(-\kappa_\sigma).$$

If one linearizes around $\kappa_\sigma = 0$, the resulting equation is

$$V = -\Delta_{\Gamma_t} \kappa_\sigma.$$

If V is replaced by an upward velocity and Δ_{Γ_t} is replaced by Δ' for the graph of w, then the equation becomes the fourth-order total variation flow if $\sigma(p) = |p'|$, i.e.,

$$w_t = -\Delta' \left(\operatorname{div}' \left(\nabla' w/|\nabla' w|\right)\right).$$

One significant property of the surface diffusion flow is the preserving property of the volume (area) enclosed by Γ_t. This is not the case for the second-order problem. However, one is able to consider a volume-preserving crystalline curvature flow, which is a nonlocal equation. For example, the volume-preserving version of (1.1) is of the form

$$V = \kappa_\sigma - \frac{1}{\mathcal{H}^{n-1}(\Gamma_t)} \int_{\Gamma_t} \kappa_\sigma d\mathcal{H}^{n-1}$$

so that $\int_{\Gamma_t} V d\mathcal{H}^{n-1} = 0$. See Section 9.1 for more discussion of the volume-preserving problem.

3. Polygonal flow

In this section, we consider a special class of a polygonal flow called admissible introduced by J. Taylor [156] and S. B. Angenent and M. E. Gurtin [10] for a planar purely crystalline curvature flow equation.

Admissible polygonal flow. We first introduce a special class of polygonal flows associated to a purely crystalline anisotropy σ. Let W_σ denote the Wulff shape corresponding to σ. Since the anisotropy σ is purely crystalline, W_σ is a bounded, convex polygon containing the origin as an interior point, but it does not need to be symmetric with respect to the origin since σ is

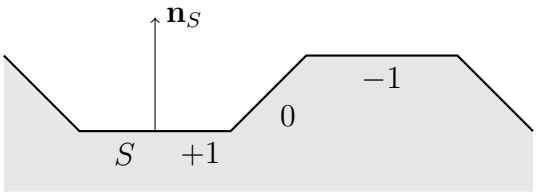

FIGURE 4. Value of χ based on the convexity/concavity near the facets.

not assumed to be even. Let $\mathcal{N}$ be a finite subset of the unit circle so that it is the set of all orientations (exterior normals) of edges on the boundary ∂W_σ of W_σ. We call $\mathcal{N}$ the set of *admissible directions*. This set can be written as

$$\mathcal{N} = \{\mathbf{n}_k\}_{k=1}^m \quad \text{with} \quad \mathbf{n}_k = (\cos\theta_k, \sin\theta_k)$$

with $0 \le \theta_1 < \cdots < \theta_m < 2\pi$. The set $\Theta = \{\theta_k\}_{k=1}^m$ is called the set of *admissible angles*, which is considered as a subset in $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For example, θ_1, θ_{m-1} are adjacent to θ_m. We say that an (oriented) polygon is *admissible* if

(i) (direction condition) the orientation of each facet (edge) is in $\mathcal{N}$;
(ii) (adjacence condition) the angles of orientations of adjacent facets should be adjacent.

An evolving polygon $\{\Gamma_t\}_{t\in I}$ is an *admissible polygonal flow* if Γ_t is an admissible polygon for $t \in I$ and the motion of all vertices is C^1 in time $t \in I$, where I is a time interval.

Crystalline curvature. Since the Wulff shape is a substitute of the unit disk, it is natural to postulate that $\kappa_\sigma = -1$ on ∂W_σ. Let $\Delta(\mathbf{n})$ denote the length of a facet (edge) of ∂W_σ whose orientation equals $\mathbf{n}$. For a general admissible polygon Γ, let S denote one of its facets. By the ansatz for curve evolution, κ_σ on S must be a constant and its value must be a kind of Cheeger ratio. In our setting on S with orientation $\mathbf{n}_S$, it is natural to assign

$$\kappa_\sigma = \chi\Delta(\mathbf{n}_S)/L,$$

where L is the length of the facet S and χ is a transition number, i.e., $\chi = +1$ (resp. -1) if Γ is convex (concave) in the direction of $\mathbf{n}_S$ near S, and otherwise $\chi = 0$; see Figure 4. By this definition, $\kappa_\sigma = -1$ on ∂W_σ since $\mathbf{n}_S$ is taken outward from W_σ; this is the outward curvature. We measure the curvature by comparing with the Wulff shape, which is consistent with the definition of the usual curvature by the inverse of the radius of the osculating circle called a circle of curvature. This quantity κ_σ is often called a *crystalline curvature*.

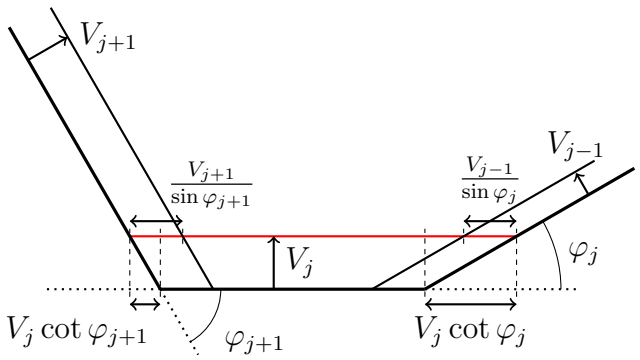

FIGURE 5. Situation near facet j with $\mathbf{n}_j$ pointing up.

Derivation of a system of ODEs. Let $\{\Gamma_t\}_{t\in I}$ be an admissible polygonal flow such that for $t \in I$, Γ_t is an ℓ-polygon consisting of facets $\{S_j(t)\}_{j=1}^{\ell}$ numbered counterclockwise and vertices of $S_j(t)$ whose motion is C^1 in time. Let $V_j(t)$ denote the normal speed of $S_j(t)$ in the direction of the orientation $\mathbf{n}_j$ of $S_j(t)$. We consider a general form of the equation

$$V = g(\nu, \kappa_\sigma) \tag{3.1}$$

with g non-decreasing in the second variable so that the problem is at least degenerate parabolic. For an admissible polygonal flow, this equation is formally reduced to

$$V_j(t) = g\left(\mathbf{n}_j, \chi_j \Delta(\mathbf{n}_j)/L_j(t)\right), \quad j = 1, \dots, \ell, \tag{3.2}$$

where $L_j(t)$ is the length of $S_j(t)$ and χ_j is the transition number of $S_j(t)$. By an elementary geometry Fig. 5, we observe that

$$\begin{aligned}\frac{dL_j(t)}{dt} =& -\frac{1}{\sin\varphi_j}V_{j-1}(t) + (\cot\varphi_j + \cot\varphi_{j+1})V_j(t)\\ &- \frac{1}{\sin\varphi_{j+1}}V_{j+1}(t), \quad j = 1, \dots, \ell,\end{aligned} \tag{3.3}$$

where $\varphi_j = \theta_j - \theta_{j-1}$ and θ_j is the angle of $\mathbf{n}_j$, i.e.,

$$\mathbf{n}_j = (\cos\theta_j, \sin\theta_j).$$

We use the convention that the indices are considered modulo ℓ, i.e., we identify $\theta_{\ell+j} = \theta_j$. We conclude (3.2) and (3.3) to get a system of ℓ ODEs for L_j's. The initial value problem of this system is locally-in-time solvable for example when g is C^1 in the second variable. The resulting admissible polygonal flow is called a *crystalline flow*. This idea is introduced by J. Taylor [156] for $V = \sigma\kappa_\sigma$ and S. B. Angenent and M. E. Gurtin [10] for $V = M(\nu)(\kappa_\sigma + C)$; both examples are introduced in Section 2.

Starting from a general polygon. If one considers a polygon whose

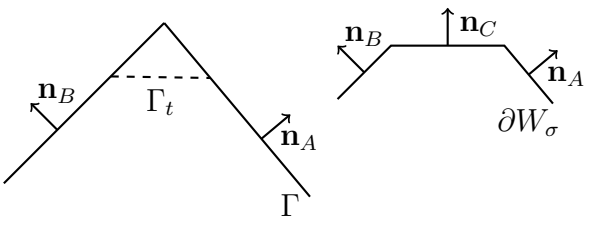

FIGURE 6. Example of an evolution of a initial polygon Γ with facets with nonadjecent orientations.

orientation belongs to $\mathcal{N}$ but violates the adjacence condition, it is expected that new facets with "missing directions" are created from a corner. To be more precise, let us consider the equation

$$V = \kappa_\sigma.$$

We consider adjacent facets S_A, S_B of a polygon Γ whose angles θ_A, θ_B of orientation $\mathbf{n}_A$, $\mathbf{n}_B$ are not adjacent; see Figure 6. In other words, there are missing admissible angles between θ_A and θ_B. If S_A and S_B do not move, i.e., they are stationary, it is expected that there is a unique self-similar expanding crystalline flow which converges to $S_A \cup S_B$ as time tends to zero. The unique existence of such a self-similar expanding crystalline flow has been claimed in a pioneering work by J. Taylor [160, Proposition 2.2 (1)]. However, unfortunately, the proof skips over important details. Here, by self-similar we mean that the flow Γ_t is of the form $\Gamma_t = t^{1/2}\Gamma_*$ with some admissible Γ_*; we here assume that the vertex connecting S_A and S_B is the origin by translation. Note that Γ_* may not be a part of the Wulff shape as observed in a numerical calculation [103]. The unique existence of such a self-similar expanding solution is proved in D. Campbell [36] in the case that W_σ is a regular polygon and in [77] for general W_σ. This problem is reduced to solving a system of algebraic equations and methods presented in [36] and [77] are quite different. Approximating by such a self-similar expanding solution, one is able to construct an expanding solution even if S_A and S_B are moving. This is carried out by Y. Ochiai [141] for $V = \kappa_\sigma$ and is extended to an equation including $V = M(\nu)(\kappa_\sigma + C)$ by R. Kuroda [124]; see also [78] for a complete proof for more general equations.

Although there is a large number of articles studying crystalline flows, this type of facet creation problems are not discussed frequently. A facet creation problem was observed in [70] and further developed in [132], [133], [134] mostly for graph-like solutions. However, the number of newly created facets in one point is just one. This aspect is quite different from works by

[160], [36], [77], [141], [124], [78], where several facets are created from one point (corner).

On the other hand, it is not difficult to handle the case when the direction condition is violated. In this case, we just regard $\Delta(\mathbf{n}) = 0$ for such directions. Such a facet is preserved at least for a short time, so we may call such a polygonal flow satisfying "adjacence condition" a *weakly admissible* polygonal flow [67].

Behavior of convex crystalline flow. If the initial polygon is a convex (admissible) polygon, the behavior of a solution (crystalline flow) has been well-studied for $V = M(\nu)\kappa_\sigma$. It is easy to see that the convexity is preserved.

(i) The case when M is parallel to σ, i.e., $M(\nu) = c\sigma(\nu)$ with some $c > 0$. It is easy to see that there always exists a self-similar solution shrinking to a point whose profile is the Wulff shape W_σ. By a spatial translation, this solution can be written as $\Gamma_t = (2c)^{1/2}(T-t)^{1/2}\partial W_\sigma$, where T is the extinction time. The uniqueness of a self-similar solution is proved when the Wulff shape W_σ is symmetric with respect to the origin and the number of its vertices is larger than four in [153]; in the case W_σ is a parallelogram, all parallelograms shrink self-similarly; see the next section for an explicit solution. Moreover, it is shown in [153] that all convex solutions shrink asymptotically similarly to the self-similar solution. These results are parallel to those for conventional curve shortening flow as established in [64], [65], [50], [49].

(ii) The case where M is unrelated to σ. In this case, the situation is complicated as discussed in [154], [9]. In [9] a rather complete picture is given. We first consider the orientation-free case i.e., $M(\mathbf{n})\Delta(\mathbf{n}) = M(-\mathbf{n})\Delta(-\mathbf{n})$ for $n \in \mathcal{N}$ where $\mathcal{N} = -\mathcal{N}$. In this case, there are two possibilities. Either phenomenon similar to (i) occurs or there is no self-similar shrinking solution and the isoperimetric ratio of a solution may tend infinity [9]. Moreover, in the second case it is shown in [9] that the minimal length of facets at time t behaves like $\{(T-t)/\log(t-t)\}^{1/2}$ or $(T-t)^\beta$, $1/2 < \beta < 1$ as t tends to T, where T is the extinction time. For a self-similar solution, the length should behave like $(T-t)^{1/2}$ so, in the second case, it is shorter than that of a self-similar solution. This has a strong contrast compared to the conventional orientation-free anisotropic curvature flow, where all flows shrink in a self-similar way. This indicates that a qualitative

property of a solution may differ from the conventional curve shorting equation depending upon the Wulff shape. If the motion is not orientation-free, it is shown in [115] that a crystalline flow may not become convex. There also exists a non-convex self-similar shrinking solution when the Wulff shape is a square or a regular triangle for $V = M\kappa_\sigma$ with M unrelated to σ which is not orientation-free [115].

We next consider the equation $V = |\kappa_\sigma|^{\alpha-1}\kappa_\sigma$ for $\alpha > 0$. The situation depends on the value of α. We have discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [72] that there is no degenerate pinching at the extinction time T. By degenerate pinching we mean that two parallel facets touch with positive length at the extinction time. For $\alpha > 1$, all (convex) solutions shrink to a point in a self-similar way like (i) [9]. If $\alpha < 1$, a degenerate pinching may happen and there is a solution whose enclosed area tends to zero but the limit of the length remains positive [9]. For $\alpha < 1$, there also exists a non-convex self-similar solution for $V = M|\kappa_\sigma|^{\alpha-1}\kappa_\sigma$ even if the equation is orientation-free [115].

We now consider the case when the initial polygon does not fulfill the direction condition but satisfies the adjacency condition with interpretation that $\Delta(\mathbf{n}_i) = 0$ for a non-admissible direction of the initial polygon and that $\mathbf{n}_i$ belongs to $\mathcal{N}$. In [169] a quite general results are established. The equation considered there is $V = g(\nu, \kappa_\sigma)$ with $g(\nu, 0) = 0$ which is non-decreasing and locally Lipschitz in the second variable. By solving the system of ODEs, we see that the number of facets is unchanged during a short time. At some time either at least one of the facets with a non-admissible direction disappears or the whole evolution shrinks to a point [169].

Behavior of a general admissible polygon. If the initial polygon is admissible but not convex, it must have an inflection facet, i.e., a facet with $\chi = 0$. There is a crystalline flow with such initial data until the length of some facet tends to zero. It is already proved in [160, Theorem 3.2] that for the equation $V = \sigma\kappa_\sigma$, at such occasion only at most two adjacent inflection facets disappear unless the flow shrinks to a point. However, the proof there is rather sketchy. In [107] a full proof is given when W_σ is a regular polygon with an even number of facets. The resulting polygon at the time when infection facets disappear stays admissible, so one can extend a solution as a crystalline flow until it loses another facet. We are able to complete this procedure until it shrinks to a point. Such an extended flow is called an extended crystalline flow.

For the curve shortening equation $V = \kappa$, it is shown that the solution (flow) becomes convex in finite time [98]. It seems that the corresponding result is not established even when W_σ is symmetric with respect to the origin and the equation is $V = \sigma\kappa_\sigma$. To the best of our knowledge, the (extended) crystalline flow (after losing several inflection facets) becomes almost convex in the sense that all facets have positive crystalline curvature possibly except two adjacent inflection facets for $V = \sigma\kappa_\sigma$ with symmetric W_σ as shown in [108].

Equations with a driving force term. We next consider the equation $V = M(\nu)(\kappa_\sigma + C)$, where C is a constant. This equation is sometimes called the eikonal curvature flow equation. There are several new phenomena in this equation compared to the case $C = 0$. For example, this motion certainly depends on the orientation. If C is taken positive and ν is taken outwards, it grows to the whole plane in infinite time provided that the polygon is sufficiently large. The large time behavior of an (extended) crystalline flow is studied in [75] with special emphasis on the anisotropic effect of mobility M and σ. For V-shaped initial data, its evolution was studied in [110], [111]. A crystalline flow is also applied to the study of a growth of spirals since the work of [112], which is further developed in [113]. Various methods for the numerical computation of the crystalline flow are compared in [114].

A few remarks on consistency and stability. If the initial data is given as the graph of a periodic piecewise linear function, for $V = M(\nu)\kappa_\sigma$ the theory of maximal monotone operators applies to construct a solution [63]. This notion of a solution is consistent with the (extended) crystalline flow; see also [52] where a numerical scheme based on a variational inequality is given.

Note that the crystalline flow satisfies a comparison principle or an order preserving property. It reads that if an admissible polygon Γ^a is enclosed by another admissible polygon Γ^b, then the corresponding crystalline flows $\{\Gamma^a_t\}$ and $\{\Gamma^b_t\}$ starting from Γ^a and Γ^b, respectively, have the same property, i.e., $\{\Gamma^b_t\}$ encloses $\{\Gamma^a_t\}$ as far as both exist; see [160] and [85]. This is easily seen by comparing their crystalline curvatures. Based on this property, one is able to establish a notion of viscosity solutions. This was first introduced in the case where Γ_t is given as the graph of a periodic function [70]; see also [68] and its consistency with an (extended) crystalline flow already discussed in [67]. Moreover, their solution can be obtained as a limit of a smoother problem, i.e., the problems where σ is smooth [71].

This stability property applies for a variational solution [63]. Note also that in both frameworks solutions with a smooth σ can be approximated by a crystalline flow [63], [71]. This gives a numerical algorithm to solve a smooth anisotropic curvature flow or even the heat equation by approximating it by crystalline flows. This topic is studied in [63], [96], [71] for a graph-like solution. In [96] a convergence rate is also given. The approach by viscosity solution is extended to closed curves through a level-set method [73] and its consistency is discussed in [72]. The stability is also discussed in [73]. Among other results, a solution with a smooth σ can be approximated by extended crystalline flows. It is proved for $V = \kappa$ in [95] for convex curves with convergence rate and in [107] for a general curve. In [73] such stability is discussed for a general equation $V = g(\nu, \kappa_\sigma)$. Note that it is also shown in [73] that an extended crystalline flow is a limit of flows of problems with smooth anisotropy. More precisely, if W_σ is close in the sense of the Hausdorff distance, the solution must be close (up to fattening).

When one discusses consistency for equations with driving force term like the eikonal-curvature flow $V = M(\nu)(\kappa_\sigma + C)$, one should be careful to handle corners. If we consider just the eikonal equation $V = C > 0$ for a bounded convex polygon, it is expected that the solution will be rounded following the Huygens principle. To preserve corners, one has to restrict the mobility $M(\nu)$. We consider a general equation $V = g(\nu, \kappa_\sigma)$. Let us explain the corner preserving condition explicitly stated in [75, Lemma 4.1, Lemma 4.2]. We say that g satisfies the *corner preserving condition* if for each $\mathbf{n}_k \in \mathcal{N}$

$$g(\mathbf{m}, 0) = \frac{1}{\sin \varphi_{k+1}} \left(g(\mathbf{n}_k, 0) \sin \psi_{k+1} + g(\mathbf{n}_{k+1}) \sin \psi_k \right)$$

for all $\mathbf{m} = (\cos\theta, \sin\theta)$ with $\theta_k < \theta < \theta_{k+1}$, where $\varphi_{k+1} = \theta_{k+1} - \theta_k$ and ψ_k (resp. ψ_{k+1}) is the angle between $\mathbf{n}_k$ ($\mathbf{n}_{k+1}$) and $\mathbf{m}$ so that $\varphi_{k+1} = \psi_{k+1} + \psi_k$. Geometrically speaking, this condition can be written as

$$A_k \subset \left\{ x \in \mathbb{R}^2 \mid x \cdot \mathbf{m} \le g(\mathbf{m}, 0),\ \mathbf{m} = (\cos\theta, \sin\theta),\ \theta_k < \theta < \theta_{k+1} \right\} \subset B_k$$

with

$$A_k = H_k \cap H_{k+1},\ B_k = H_k \cup H_{k+1},\ H_{k+j} = \left\{ x \in \mathbb{R}^2 \mid x \cdot \mathbf{n}_{k+j} \le g(\mathbf{n}_{k+j}, 0) \right\}.$$

If Γ is convex with outward orientation, we only need the inclusion of A_k. In other words, in the above identity the equality should be replaced by $\ge$ so that $g(\mathbf{m}, 0)$ is always larger than the right-hand side. This condition says that in the corner all segments whose orientation is between that of facets forming the corner move faster than corner facets for $V = g(\nu, 0)$.

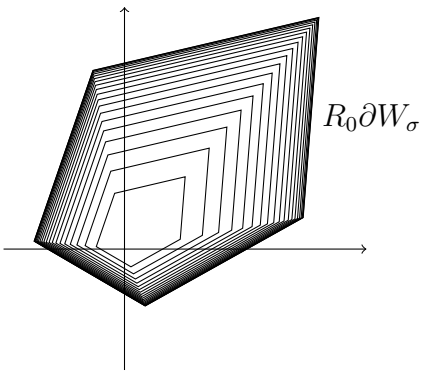

FIGURE 7. Timesteps $\Gamma_{it^*/20}$, $i = 0, 1, \ldots$, of a homethetic Wulff shape solution of $V = \sigma\kappa_\sigma$. Note that even though κ_σ is a constant on Γ_t, the edges further from the origin move faster due to the mobility factor σ and the solution is homothetic.

This condition is first pointed out explicitly by [82] and independently by [100]. It is stated in a different from in [67]. The geometric version is found in [72]; however, unfortunately, the definition of B_k was mistyped.

We shall postpone the definition of viscosity solutions to Section 6. We note that the theory covers a wide range of σ not necessarily purely crystalline in planar case for general equation $V = g(\nu, \kappa_\sigma)$ including (2.1), (2.2), while in higher dimension, for such setting it is limited for purely crystalline σ for general equations; see Section 6. In [73] it is only assumed that F_σ is convex, C^2 except finitely many vertices and the curvature is bounded.

Although the approach by admissible polygonal flow is convenient to study planar curvature flow equations, it is limited because it implicitly requires that the speed of a facet is spatially constant. For example, even in $\mathbb{R}^2$ if one considers the equation with spatially inhomogeneous driving force like

$$V = M(\nu)\left(\kappa_\sigma + f(x)\right),$$

then it is not appropriate to assign the speed of a facet as a spatially constant to obtain a comparison principle. For a graph-like solution with special M, as a variational solution several facet-breaking solutions are given in [69].

4. Explicit solutions

In this section we give examples of a few interesting explicit solutions to illustrate the behavior of the equations.

The simplest solution of the crystalline mean curvature flow is the homothetic (self-similar) solution starting from the Wulff shape, Fig. 7, that can be translated and scaled. Rotations are of course not allowed. As noted in

the introduction, the crystalline mean curvature on the surface of the Wulff shape W_σ is the constant $n-1$; here, the orientation (normal) is taken inward. Therefore

$$\Omega_t = \sqrt{R_0^2 - 2(n-1)t}\, W_\sigma$$

is a solution of the crystalline mean curvature flow $V = \sigma(\nu)\kappa_\sigma$ for any $R_0 > 0$ on the interval $t \in [0, t^*)$, where $t^* = \frac{R_0^2}{2(n-1)}$ is the *extinction time*. Note the factor $\sigma(\nu)$ in the velocity law. The (inner) normal velocity of $R(t)W_\sigma$ at a boundary point x with inner unit normal ν is $-R'(t)x \cdot \nu = -R'(t)\sigma(\nu)R(t)$.

One might ask whether the above solutions are the only homothetic solutions of the flow. This is however not always the case as the following simple example illustrates. We consider $n = 2$ and the anisotropy $\sigma(p) = |p_1| + |p_2| = \|p\|_1$. Let $\Omega_0 = (-a, a) \times (-b, b)$ be a rectangle for some $a > 0$, $b > 0$. Then $\Omega_t = R(t)\Omega_0$ for $R(t) = \sqrt{1 - \frac{2}{ab}t}$ is a solution of both $V = \sigma\kappa_\sigma$ and $V = \kappa_\sigma$. In $n = 2$, the uniqueness of the Wulff shape homothetic solution was proved by Stancu [153] for even anisotropies σ when (so that the problem is orientation-free) the Wulff shape W_σ is not a quadrilateral as mentioned in Section 3 (i).

A related question is whether a solution starting from an arbitrary convex initial data will asymptotically approach the homothetic Wulff shape solution as in the case of the usual mean curvature flow. As mentioned in Section 3 (i), this was shown again by Stancu [154] in $n = 2$ for even non-quadrilateral anisotropies. The situation is much more complex in $n > 2$ and is studied in [139].

By an interpretation different from Section 3 (ii), we also mention that for $V = \sigma\kappa_\sigma$ examples of non-convex homothetic solutions in $n = 2$ given in [115] for non-even anisotropies σ, that is, $\sigma(p) \neq \sigma(-p)$ for some p. This shows that one cannot in general expect that a non-convex connected initial shape will become convex before the extinction time.

In dimensions $n > 2$ the situation is more complex and nonzero genus explicit homothetic solutions are known. For example, for $\sigma(p) = \|p\|_1$ a cube with a square-shaped hole along each axis is a homothetic solution, Figure 8. See [148] for more details.

For examples of solutions of the related total variation flow see for example [130, Sec. 5].

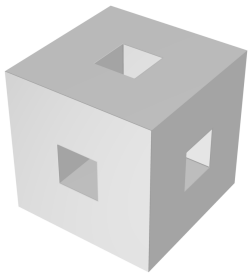

FIGURE 8. Sponge-like homothetic solution of the crystalline mean curvature flow in dimension 3 with $\sigma(p) = \|p\|_1$.

5. Approach by the theory of maximal monotone operators

5.1. **Abstract theory.** In this section we introduce the crystalline mean curvature as a solution of a certain minimization problem. This interpretation is based on the theory of maximal monotone operators of Kōmura [123] and Brezis [31].

Let us give a motivation for this point of view. It is natural to expect that the crystalline mean curvature flow with anisotropy σ can be approximated by anisotropic mean curvature flow with smooth anisotropies σ_m so that $\sigma_m \to \sigma$ in some sense.

If Γ is a C^2 surface in $\mathbb{R}^n$ and σ_m is a C^2 smooth anisotropy, the anisotropic mean curvature Γ at $x \in \Gamma$ is given as $\operatorname{div} \nabla\sigma_m(\nabla u)(x) = \operatorname{tr}[\nabla^2\sigma_m(\nabla u(x))\nabla^2 u(x)]$, where u is any C^2 level set function of Γ in the neighborhood of x with $\nabla u(x) \neq 0$.

Consider now the anisotropic total variation flow

$$u_t - \operatorname{div} \nabla\sigma_m(\nabla u) = 0$$

on $L^2(\mathbb{T}^n)$, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, or more rigorously,

$$u_t \in -\partial\mathcal{E}_m(u), \tag{5.1}$$

where

$$\mathcal{E}_m(v) := \begin{cases} \int_{\mathbb{T}^n} \sigma_m(\nabla v)\, dx & v \in BV(\mathbb{T}^n) \cap L^2(\mathbb{T}^n), \\ +\infty & \text{otherwise}, \end{cases} \tag{5.2}$$

is the anisotropic total variation energy.

Since $\mathcal{E}_m$ is a convex lower semicontinuous functional on the Hilbert space $L^2(\mathbb{T}^n)$ with a dense domain, the operator $\partial\mathcal{E}_m$ is a maximal monotone operator and the anisotropic total variation flow (5.1) has a unique solution for any initial data in $L^2(\mathbb{T}^n)$.

Suppose now that $(\sigma_m)_{m\geq 1}$ is a sequence of C^2 anisotropies that monotonically converge to the crystalline anisotropy σ. Then it is known [12] that

$\mathcal{E}_m \to \mathcal{E}$ in the sense of Mosco convergence, see (1.7), where

$$\mathcal{E}(v) := \begin{cases} \int_{\mathbb{T}^n} \sigma(\nabla v)\, dx & v \in BV(\mathbb{T}^n) \cap L^2(\mathbb{T}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

But Mosco convergence implies the convergence of the nonlinear semigroups [12, Theorem 3.26] and [32, Theorem 3.2]: the solutions of (5.1) locally uniformly converge to the unique solution $u : [0,\infty) \to L^2(\mathbb{T}^n)$ of

$$\begin{cases} \dfrac{du}{dt} \in -\partial\mathcal{E}(u(t)), & t > 0, \\ u(0) = u_0 \end{cases} \tag{5.3}$$

for any initial data $u_0 \in L^2(\mathbb{T}^n)$.

As we will see below, $\partial\mathcal{E}(v)$ is in general multivalued even if $\nabla v \neq 0$ for typical crystalline mean curvature evolutions. Nevertheless, the unique solution of (5.3) is right-differentiable at every $t > 0$, $\partial E(u(t)) \neq \emptyset$ and $d^+u/dt(t) = -\partial^0\mathcal{E}(u(t))$ for $t > 0$, where $\partial^0\mathcal{E}(v)$ is the *canonical restriction* or *minimal section* of the subdifferential $\partial\mathcal{E}(v)$, i.e., the unique element of $\partial\mathcal{E}(v) \subset L^2(\mathbb{T}^n)$ with minimal norm.

This strongly suggests that we should use $-\partial^0\mathcal{E}(v)$ as the definition of the crystalline mean curvature to hope to obtain stability under approximation by anisotropic mean curvature flow.

Fortunately the characterization of $\partial\mathcal{E}$ is well understood even for rather general $\sigma = \sigma(x,p)$, see [130] for example.

We include the proof here for completeness for $\sigma = \sigma(p)$ and space $L^2(\mathbb{T}^n)$. We need to introduce a number of definitions.

The functional $\mathcal{E}(u)$ can be defined in two equivalent ways. The first one is a generalization of the definition of the total variation,

$$\mathcal{E}(u) := \sup\left\{ -\int_{\mathbb{T}^n} u \operatorname{div} z\, dx \;\middle|\; \sigma^\circ(z) \le 1,\ z \in C^1(\mathbb{T}^n, \mathbb{R}^n) \right\}, \qquad u \in L^2(\mathbb{T}^n).$$

Note the minus sign since σ is not assumed even. The function σ° is the support function of the Frank diagram $F_\sigma = \{\sigma \le 1\}$, i.e.,

$$\sigma^\circ(x) = \sup\{x \cdot p \mid p \in F_\sigma\}$$

so that $W_\sigma = \{\sigma^\circ \le 1\}$. This $\mathcal{E}$ is clearly a convex, positively one-homogeneous, lower semi-continuous functional on $L^2(\mathbb{T}^n)$. It is known [5] that it is the *relaxation* (*closure* or lower semicontinuous envelope) of the functional

$$J(u) := \begin{cases} \int_{\mathbb{T}^n} \sigma(\nabla u)\, dx, & u \in W^{1,1}(\mathbb{T}^n) \cap L^2(\mathbb{T}^n), \\ +\infty, & \text{otherwise.} \end{cases}$$

In fact $\mathcal{E}(u) = J(u)$ for any $u \in W^{1,1}(\mathbb{T}^n) \cap L^2(\mathbb{T}^n)$. While [5] deals with L^1 relaxation, it is not difficult to see that $\mathcal{E}$ is also the L^2 relaxation. Indeed,

we have $\mathcal{E}(u) = \int_{\mathbb{T}^n} \sigma(\frac{Du}{|Du|})\, d|Du|$ for any $u \in BV(\mathbb{T}^n)$ and $\mathcal{E}(u) = +\infty$ for $u \in L^2(\mathbb{T}^n) \setminus BV(\mathbb{T}^n)$, where $\frac{Du}{|Du|}$ is the Radon-Nikodym derivative of the Radon measure Du with respect to its total variation [130]. Then by standard approximation of $u \in BV(\mathbb{T}^n) \cap L^2(\mathbb{T}^n)$ there is a sequence $u_k \in W^{1,1}(\mathbb{T}^n) \cap L^2(\mathbb{T}^n)$ converging to u in L^2 such that $|Du_k| \to |Du|$. For such sequence we have $\mathcal{E}(u) = \lim_{k\to+\infty} \mathcal{E}(u_k) = \lim_{k\to+\infty} J(u_k)$ by Reshetnyak's continuity theorem [149]. We conclude that $\mathcal{E}$ is the L^2 relaxation of J.

To characterize the subdifferential, we here present a simplification of the proof in [130], which itself is based on the unpublished note of F. Alter; see also [7] for the proof when $\sigma(p) = |p|$ and more details. The idea is based on the characterization of the subdifferential using the polar of $\mathcal{E}$ defined as

$$\mathcal{E}^\circ(v) := \sup\{(u,v) \mid u \in H,\ \mathcal{E}(u) \le 1\} = \sup\left\{ \frac{(u,v)}{\mathcal{E}(u)} \middle| u \in H \right\},$$

where we set $H := L^2(\mathbb{T}^n)$ the Hilbert space with the L^2-inner product $(u,v) = \int uv\, dx$. In the formula we use the convention $\frac{0}{0} = 0$, $\frac{a}{0} = +\infty$ for any $a > 0$ and $\frac{a}{+\infty} = 0$ for any $a \in \mathbb{R}$. Since $\mathcal{E}$ is positively one-homogeneous, convex and lower semicontinuous, we have the following standard characterization [7, Lemma 1.7]:

$$v \in \partial\mathcal{E}(u) \qquad \Leftrightarrow \qquad \mathcal{E}^\circ(v) \le 1 \text{ and } (v,u) = \mathcal{E}(u). \tag{5.4}$$

We will show that $\mathcal{E}^\circ$ coincides with the functional

$$\Psi(v) := \inf\{\|\sigma^\circ(z)\|_\infty \mid v = -\operatorname{div} z,\ z \in L^\infty(\mathbb{T}^n)\}, \qquad v \in H. \tag{5.5}$$

The equality $v = -\operatorname{div} z$ is understood in the sense of distributions: the function $-v$ is the distributional divergence of z. For the discussion of properties of Ψ, see Lemma A.1 in Appendix A below. In particular, Ψ is again a convex, positively one-homogeneous, lower semicontinuous functional. For any such functional we have $(\Psi^\circ)^\circ = \Psi$ [7, Proposition 1.6]. Moreover, if $\Psi(v) < \infty$ the infimum is attained by a vector field and hence it is a minimum.

Theorem 5.1. *The equality $\mathcal{E}^\circ \equiv \Psi$ holds.*

Proof. $\boxed{\le}$: Take $v \in H$ with $\Psi(v) < \infty$ and fix $z \in L^\infty(\mathbb{T}^n)$ with $\operatorname{div} z = -v$. Since $\mathcal{E}$ is the lower semicontinuous envelope of J, for any $u \in H$ with $\mathcal{E}(u) \le 1$ there is a sequence $(u_k)_k \subset W^{1,1}(\mathbb{T}^n) \cap H$ with $u_k \to u$ in H and $J(u_k) = \mathcal{E}(u_k) \to \mathcal{E}(u)$. We have

$$\begin{aligned}(u_k, v) &= \int z \cdot \nabla u_k\, dx \le \int \sigma^\circ(z)\sigma(\nabla u_k)\, dx \\ &\le \|\sigma^\circ(z)\|_\infty \int \sigma(\nabla u_k)\, dx = \|\sigma^\circ(z)\|_\infty\, \mathcal{E}(u_k).\end{aligned}$$

In the limit $k \to \infty$ we obtain

$$(u, v) \le \|\sigma^\circ(z)\|_\infty \qquad \text{for all } u \in H \text{ with } \mathcal{E}(u) \le 1.$$

Thus by definition of Ψ we deduce $\mathcal{E}^\circ(v) \le \Psi(v)$.

$\boxed{\ge}$: Fix $u \in H$. By definition we have

$$\begin{aligned}\mathcal{E}(u) &= \sup\left\{ -\int u \operatorname{div} z \, dx \;\middle|\; \sigma^\circ(z) \le 1,\ z \in C^1(\mathbb{T}^n, \mathbb{R}^n) \right\} \\ &= \sup_{z \in C^1} \frac{(u, -\operatorname{div} z)}{\|\sigma^\circ(z)\|_\infty} \le \sup_{z \in C^1} \frac{(u, -\operatorname{div} z)}{\Psi(-\operatorname{div} z)} \\ &\le \Psi^\circ(u),\end{aligned}$$

where we again use $0/0 := 0$. We deduce $\mathcal{E}^\circ \ge (\Psi^\circ)^\circ = \Psi$. □

We now have the following characterization of the subdifferential for Lipschitz functions. For the general characterization we refer the reader to Theorem 12 in [130].

Corollary 5.2. *Let $u \in Lip(\mathbb{T}^n)$ and $v \in L^2(\mathbb{T}^n)$. The following are equivalent:*

- $v \in \partial\mathcal{E}(u)$
- *there exists $z \in L^\infty(\mathbb{T}^n)$ with $v = -\operatorname{div} z$ such that $z \in \partial\sigma(\nabla u)$ a.e.*

Proof. $\boxed{\Rightarrow}$: $v \in \partial\mathcal{E}(u)$ implies that $\mathcal{E}^\circ(v) \le 1$ and $\mathcal{E}(u) = (u, v)$. In particular by Theorem 5.1 there exists a vector field $z \in L^\infty(\mathbb{T}^n)$ with $v = -\operatorname{div} z$ and $\|\sigma^\circ(z)\| = \mathcal{E}^\circ(v) \le 1$. We have

$$\nabla u \cdot z \le \sigma(\nabla u)\sigma^\circ(z) \le \sigma(\nabla u) \qquad a.e.$$

However, $\mathcal{E}(u) = (u, v)$ and therefore

$$\int \sigma(\nabla u)\, dx = \int uv \, dx = \int \nabla u \cdot z \, dx,$$

and so we can deduce that $\nabla u \cdot z = \sigma(\nabla u)$ a.e., which coupled with $\sigma^\circ(z) \le 1$ a.e. implies $z \in \partial\sigma(\nabla u)$ a.e.

$\boxed{\Leftarrow}$: The opposite implication can be proved by reversing the above steps. □

The vector fields z play a central role and we define

$$X^2(U) := \{ z \in L^\infty(U) \mid \operatorname{div} z \in L^2(U) \},$$

for $U \subset \mathbb{R}^n$ open or $U = \mathbb{T}^n$, following [11]. The vector fields that characterize the subdifferential are often called *Cahn-Hoffman vector fields* and we define

$$CH(u; U) := \{ z \in X^2(U) \mid z \in \partial\sigma(\nabla u) \text{ a.e.} \} \tag{5.6}$$

for any $u \in Lip(U)$. Note that if $U = \mathbb{T}^n$, by Corollary 5.2

$$-\partial\mathcal{E}(u) = \operatorname{div} CH(u;\mathbb{T}^n) := \{\operatorname{div} z \mid z \in CH(u;\mathbb{T}^n)\}.$$

Recall that this is a closed convex set, but it might be empty.

Since the set $\partial\mathcal{E}(u)$ is in general not a singleton, we need to determine how to select a value that gives a reasonable generalization of the anisotropic mean curvature to the crystalline case. The theory of maximal monotone operators suggests that we should choose the unique element of $-\partial\mathcal{E}(u)$ with the smallest L^2-norm. We will denote this element $-\partial^0\mathcal{E}(u)$ if $\partial\mathcal{E}(u) \neq \emptyset$, since it is the projection of the origin 0 on the convex closed set $-\partial\mathcal{E}(u)$.

Example 5.3. Suppose that $\sigma \in C^2(\mathbb{R}^n \setminus \{0\})$ and $u \in C^2(\mathbb{T}^n)$. Let $x \in \mathbb{T}^n$ with $\nabla u(x) \neq 0$. Then $\partial\sigma(\nabla u) = \{\nabla\sigma(\nabla u)\}$ in the neighborhood of x and therefore if $z \in CH(u;\mathbb{T}^n)$ we necessarily have $\operatorname{div} z(x) = \operatorname{div} \nabla\sigma(\nabla u)(x)$.

As was shown in the introduction, the element $-\partial^0\mathcal{E}(u)$ is a solution of a minimization problem with an n-dimensional obstacle $z \in \partial\sigma(\nabla u)$. The value of the minimizer $\operatorname{div} z_{\min}$ can depend nonlocally on u whenever $\partial\sigma(\nabla u)$ is not a singleton, as is illustrated in the introduction. However this nonlocality is restricted to "flat" parts of u. Those correspond to facets and edges of the evolving crystal. The following technical "patching" lemma was proved in [91, Lemma 2.8]. Let $\mathbf{1}_E$ denote the characteristic function of E, i.e., $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \notin E$.

Lemma 5.4. *Let $\sigma : \mathbb{R}^n \to \mathbb{R}$ be a positively one-homogeneous convex function. Let U_1, U_2 be two open subsets of $\mathbb{R}^n$ and $\psi_i \in Lip(U_i)$ two Lipschitz functions. Let $\delta > 0$ and set $G := \{x \in U_1 \mid |\psi_1(x)| < \delta\}$. Suppose that $\overline{G} \subset U_1 \cap U_2$ and $\psi_1 = \psi_2$ on G. If $z_i \in CH(\psi_i; U_i)$ are two Cahn–Hoffman vector fields, then*

$$z := z_1\mathbf{1}_{U_1\setminus G} + z_2\mathbf{1}_G$$

is also a Cahn–Hoffman vector field $z \in CH(\psi_1; U_1)$, and

$$\operatorname{div} z = \operatorname{div} z_1\mathbf{1}_{U_1\setminus G} + \operatorname{div} z_2\mathbf{1}_G.$$

We add the following simple observation that follows from $|\{0 < |\psi_1| < \delta\}| \to 0$ as $\delta \to 0$. Note that we still need $\psi_1 = \psi_2$ on a neighborhood of $\{\psi_1 = 0\}$.

Corollary 5.5. *Under the assumptions of Lemma 5.4,*

$$z := z_1\mathbf{1}_{U_1\setminus\{\psi_1=0\}} + z_2\mathbf{1}_{\{\psi_1=0\}}$$

is also a Cahn–Hoffman vector field $z \in CH(\psi_1; U_1)$, and

$$\operatorname{div} z = \operatorname{div} z_1\mathbf{1}_{U_1\setminus\{\psi_1=0\}} + \operatorname{div} z_2\mathbf{1}_{\{\psi_1=0\}}.$$

The above lemma shows that we can isolate $\operatorname{div} z_{\min}$ on a neighborhood of $\{\psi = 0\}$. This is necessary to have some locality of the crystalline mean curvature which allows us to localize the construction of test functions to a given facet.

We conclude this section by an important way of approximating the values $\partial^0\mathcal{E}(\psi)$. Let us now fix the domain $\mathbb{T}^n$ for simplicity. For a given $\psi \in L^2(\mathbb{T}^n)$ and $a > 0$, we consider the *resolvent problem*

$$v + a\partial\mathcal{E}(v) \ni \psi \tag{5.7}$$

in the unknown $v \in L^2(\mathbb{T}^n)$. This can be viewed as the implicit Euler discretization of the gradient flow (5.3). It is also the Euler–Lagrange equation of the minimization problem

$$\operatorname*{arg\,min}_{v\in L^2(\mathbb{T}^n)} \frac{\|v-\psi\|^2_{L^2(\mathbb{T}^n)}}{a} + \mathcal{E}(v),$$

which appears in an important discrete approximation of the crystalline mean curvature flow, Chambolle's scheme discussed in Section 6.4.

We have the following standard existence and approximation result that is valid for any convex proper lower semi-continuous functional like $\mathcal{E}$, see for example [12].

Proposition 5.6. *For every $\psi \in L^2(\mathbb{T}^n)$ and $a > 0$ the resolvent problem (5.7) has a unique solution $\psi_a \in L^2(\mathbb{T}^n)$ and $\psi_a \to \psi$ as $L^2(\mathbb{T}^n)$.*

If furthermore $\partial\mathcal{E}(\psi) \neq \emptyset$, then

$$\frac{\psi_a - \psi}{a} \to -\partial^0\mathcal{E}(\psi) \qquad \text{in } L^2(\mathbb{T}^n) \text{ as } a \to 0.$$

The solutions also satisfy a comparison principle, see [38] for a proof.

Proposition 5.7. *If $\psi^1, \psi^2 \in L^2(\mathbb{T}^n)$ are two right-hand sides with $\psi^1 \le \psi^2$ and $a > 0$, we have $\psi^1_a \le \psi^2_a$ where ψ^1_a and ψ^2_a are the respective solutions of (5.7).*

5.2. **Calibrability and Cheeger sets.** As we already briefly mentioned in the introduction, the minimization problem one needs to solve to find the value $\partial^0\mathcal{E}(\psi)$ for a given ψ has interesting connections to the so-called Cheeger problem for sets. For a given open set $U \subset \mathbb{R}^n$, define the *Cheeger constant* as

$$h(U) := \inf\left\{ \frac{P(F)}{\mathcal{L}^n(F)} : F \text{ Borel} \subset U,\ \mathcal{L}^n(F) \in (0,\infty) \right\},$$

where $P(F) = \mathcal{E}(\mathbf{1}_F)$ is the anisotropic perimeter of F. Usually the isotropic $\sigma(\nu) = 1$ is considered, in which case this is just the usual perimeter equal to

$\mathcal{H}^{n-1}(\partial F)$ for sufficiently regular sets. A set $F \subset U$ such that $\frac{P(F)}{\mathcal{L}^n(F)} = h(U)$ is called a Cheeger set of U. If U itself is a Cheeger set of U, it is called *self-Cheeger*. Finding the value $h(U)$ or characterizing the Cheeger subsets of U is then often referred to as the Cheeger problem. For a recent review of this topic see [126].

In the current note, the question whether a given set U is self-Cheeger is closely related to the questions whether the value of $\partial^0\mathcal{E}(\psi)$ is constant on a facet $\overline{U}$ of ψ. If $\partial^\circ\mathcal{E}(\psi)$ is constant on a given facet, the facet is called *calibrable* or *σ-calibrable*, see [26] and also [20].

We point out that this notion of calibrability is slightly weaker than the notion used in the context of total variation flows [3, 126]. There an open bounded set U is called calibrable if the total variation flow (5.3) with initial data $\mathbf{1}_U$ has the unique solution $a(t)\mathbf{1}_U$ with $a(t) = \max(1 - \frac{P(U)}{\mathcal{L}^n(U)}t, 0)$. This therefore implies that $\partial^0\mathcal{E}(\mathbf{1}_U)$ is constant on U *and* on U^{c}.

We use the former notion of calibrability. The following theorem in a more general setting (but still only in dimension $n = 2$), including non-uniform forcing, was proved in [26, Th. 6.1]. See also [6] for further developments.

Theorem 5.8. *Let $n = 2$ and let σ be an even anisotropy on $\mathbb{R}^2$, $\sigma(p) = \sigma(-p)$. Suppose that $\psi \in Lip(\mathbb{R}^2)$ is such that $CH(\psi;\mathbb{R}^2)$ is nonempty. Let U be a bounded connected component of* $\operatorname{int}\{\psi = 0\}$*. The following are equivalent:*

(i) *U is calibrable ($\partial^0\mathcal{E}(\psi)$ is constant on U)*
(ii) *for any $F \subset U$ of finite perimeter*

$$\frac{SP(F)}{\mathcal{L}^2(F)} \geq \frac{SP(U)}{\mathcal{L}^2(U)}. \tag{5.8}$$

Here $SP(F)$ is the signed perimeter of F defined using the reduced boundary $\partial^* F$ as

$$SP(F) = \int_{\partial^* F_+} \sigma(\nu)\, d\mathcal{H}^1 - \int_{\partial^* F_-} \sigma(\nu)\, d\mathcal{H}^1,$$

with $\partial^* F_- := \{x \in \partial^* F \cap \partial^* U : \nu_U(x) \cdot \nabla\psi(x) < 0\}$ and $\partial^* F_+ := \partial^* F \setminus \partial^* F_-$.

The quantity $\frac{SP(U)}{\mathcal{L}^n(U)}$ is a generalization of the usual Cheeger ratio $\frac{P(U)}{\mathcal{L}^n(U)}$ to facets: sets with signed boundary whose sign is determined by whether the surface at the boundary point is convex or concave in the normal direction of the facet; see also Section 5.3 below for a notion of facet.

To illustrate the proof of (i) $\Rightarrow$ (ii) in a simplified setting, consider now a Lipschitz function $\psi \in Lip(\mathbb{R}^n)$ whose int $\{\psi = 0\}$ is a simply connected bounded open set $U \subset \mathbb{R}^n$ with Lipschitz boundary. Let us also for simplicity assume that we can define $\nabla\psi \neq 0$ on ∂U $\mathcal{H}^{n-1}$-a.e. as the limit of $\nabla\psi$ from $\overline{U}^{\mathsf{c}}$. Suppose that there exists a vector field $z \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ with $\operatorname{div} z \in L^2(\mathbb{R}^n)$ and $z \in \partial\sigma(\nabla\psi)$ a.e. that is sufficiently regular so that the following calculation can be justified and assume that $\operatorname{div} z = \lambda$ on U for some λ. Then the divergence theorem yields

$$\lambda\mathcal{L}^n(U) = \int_U \operatorname{div} z \, dx = \int_{\partial U} z \cdot \nu \, \mathcal{H}^{n-1}.$$

We observe that $\nu = \frac{\nabla\psi}{|\nabla\psi|}$ on ∂U_+ and $\nu = -\frac{\nabla\psi}{|\nabla\psi|}$ on ∂U_-. Since $z \in \partial\sigma(\nabla\psi)$, we have $z \cdot \nu = \pm\sigma(\nu)$ on $\partial U_\pm$. We have

$$\int_{\partial U} z \cdot \nu \, \mathcal{H}^{n-1}. = \int_{\partial U_+} \sigma(\nu) \, d\mathcal{H}^{n-1} - \int_{\partial U_-} \sigma(\nu) \, d\mathcal{H}^{n-1} = SP(U).$$

In particular, $\lambda = \frac{SP(U)}{\mathcal{L}^n(U)}$. However, for any Lipschitz subset F of U we have

$$\int_F \operatorname{div} z \, dx = \int_{\partial F} z \cdot \nu \, d\mathcal{H}^{n-1} = \int_{\partial F \setminus \partial U} z \cdot \nu \, d\mathcal{H}^{n-1} + \int_{\partial F \cap \partial U} z \cdot \nu \, d\mathcal{H}^{n-1}.$$

Using the estimate

$$\int_{\partial F \setminus \partial U} z \cdot \nu \, d\mathcal{H}^{n-1} \leq \int_{\partial F \setminus \partial U} \sigma^\circ(z)\sigma(\nu) \, d\mathcal{H}^{n-1} \leq \int_{\partial F \setminus \partial U} \sigma(\nu) \, d\mathcal{H}^{n-1},$$

we deduce that $\frac{SP(F)}{\mathcal{L}^n(F)} \geq \lambda = \frac{SP(U)}{\mathcal{L}^n(U)}$.

However, it seems that the proof of (ii) $\Rightarrow$ (i) in Theorem 5.8 is available only for $n = 2$. We expect it to be valid in arbitrary dimension.

Let us give a well-known example of a facet that breaks immediately in the evolution.

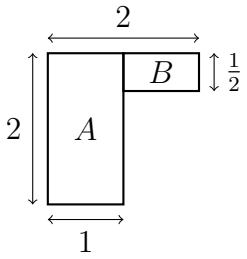

FIGURE 9.

Example 5.9. Let $n = 2$ and $\sigma(p) = \|p\|_1 = |p_1| + |p_2|$. Consider the set $C = A \cup B$ with $A = [-1, 0] \times [-1, 1]$ and $B = [0, 1] \times [\frac{1}{2}, 1]$, see Figure 9, and let $\psi(x) = \operatorname{dist}(x, C)$. It is well-known that C considered as a facet of

ψ is not calibrable and breaks into two facets A and B moving at different speeds. See [25] for the computation in the crystalline flow case and [130, Sec. 5] for the explicit computation in the anisotropic total variation flow case. [130] shows that the solution of the anisotropic total variation flow (5.3) with initial data $u_0 = \mathbf{1}_C$ is given as

$$u(x,t) = \max(1-3t,0)\mathbf{1}_A + \max(1-4t,0)\mathbf{1}_B.$$

Let us set $U = \operatorname{int} C$. In terms of Theorem 5.8 note that $SP(U) = P(U) = 8$ and $\mathcal{L}^2(U) = \frac{5}{2}$, yielding a Cheeger ratio $SP(U)/\mathcal{L}^2(U) = \frac{16}{5} = 3 + \frac{1}{5}$, while A has a Cheeger ratio $SP(A)/\mathcal{L}^2(A) = \frac{6}{2} = 3$, violating (5.8). U therefore cannot be calibrable.

5.3. **Curvature-like quantity.** The characterization of the subdifferential of the anisotropic total variation and the localization of the canonical restriction $-\partial^0\mathcal{E}$ motivates the following definition of the crystalline mean curvature. To allow for a forced mean curvature flow, we need to include the forcing into the definition. We follow the notation in [93].

Suppose that $U \subset \mathbb{R}^n$ is an open set and $\psi \in Lip(U)$. If $CH(\psi;U)$ defined in (5.6) is nonempty we define the σ°-(L^2) divergence of ψ for any $f \in L^2(U)$ as

$$\Lambda_f[\psi] := \operatorname{div} z_{\min} - f \qquad \text{on } \{\psi = 0\}$$

where $z_{\min}$ is a minimizer of $\|\operatorname{div} z - f\|_{L^2}(U)$ on $CH(\psi;U)$, that is, $\operatorname{div} z_{\min}$ is the projection of f onto $\operatorname{div} CH(\psi;U)$. Since $\operatorname{div} CH(\psi;U)$ is closed convex, the value $\operatorname{div} z_{\min}$ is unique, but $z_{\min}$ might not be.

One might wonder whether the value of $\Lambda_f[\psi]$ depends on the choice U, but thanks to the patching Lemma 5.4 that is not the case. For details see [91, Prop. 4.10].

Remark 5.10. Note that since $\partial\sigma$ is positively 0-homogeneous, $\Lambda_f[t\psi]$ does not depend on $t > 0$, and in fact for any Lipschitz function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta(0) = 0$ and $\theta'(s) > 0$ for a.e. s we have $\Lambda_f[\theta \circ \psi] = \Lambda_f[\psi]$. Indeed, by the chain rule for the Lipschitz functions $\nabla(\theta \circ \psi)(x) = \theta'(\psi(x))\nabla\psi(x)$ a.e. if we interpret the right-hand side as 0 when $\nabla\psi = 0$. The 0-homogeneity of $\nabla\sigma$ implies that $CH(\psi;U) = CH(\theta \circ \psi;U)$.

We also note the scaling invariance

$$\Lambda_f[\psi](x) = a^{-1}\Lambda_{af(a\cdot)}[\psi(a\cdot)](ax),$$

thanks to which we can always assume that $U \subset (-\frac{1}{2},\frac{1}{2})^n$.

In general, Λ_0 is only BV and can be discontinuous as was shown in [27], [28]. Finding the value of $\Lambda_f[\psi]$ explicitly in dimensions $n \geq 2$ is in general

difficult. However, if $\{\psi = 0\}$ has a sufficiently regular boundary and there is a vector field in $CH(\psi; U)$ with constant divergence on $\{\psi = 0\}$, then $\Lambda_f[\psi]$ can be found as the ratio of the signed anisotropic perimeter and the area (volume) of the facet. Such facets are referred to as *calibrable*. Even though this is well-known in the literature, we have not found a statement that applies precisely to our setting and therefore we present it here with a proof.

Lemma 5.11. *Let $U \subset \mathbb{R}^n$ be bounded open set. Suppose that $\psi \in Lip(U)$, $|\psi| > 0$ on ∂U and there exists $\delta_0 > 0$ such that $|\nabla\psi| > 0$ a.e. on $\{0 < |\psi| < \delta_0\}$ and the sets $\{\psi < \delta\}$, $\{-\psi < \delta\}$ are Lipschitz regular for $\delta \in (0, \delta_0)$, and*

$$\int_{\partial\{\pm\psi<\delta\}} \sigma(\pm\nu)\, d\mathcal{H}^{n-1} \to \int_{\partial\{\pm\psi<0\}} \sigma(\pm\nu)\, d\mathcal{H}^{n-1} \qquad \text{as } \delta \to 0,$$

where ν is the outer unit normal to the respective sets. If there exists $z_C \in CH(\psi; U)$, that satisfies $\operatorname{div} z_C = C$ a.e. on $\{\psi = 0\}$ for some constant $C \in \mathbb{R}$, then

(5.9)

$$\Lambda_0[\psi] = C = \frac{\int_{\partial\{\psi\le 0\}} \sigma(\nu)\, d\mathcal{H}^{n-1} - \int_{\partial\{\psi\ge 0\}} \sigma(-\nu)\, d\mathcal{H}^{n-1}}{|\{\psi = 0\}|} \qquad \text{a. e. on } \{\psi = 0\}.$$

If ψ is non-positive and $\sigma = 1$, then this number C is the Cheeger ratio of the set $\{\psi = 0\}$ if the boundary $\partial\{\psi \ge 0\}$ is Lipschitz. Note that we invoke only approximability of surface energy by that of Lipschitz regular set and do not assume Lipschitz regularity of $\partial\{\psi \ge 0\}$ itself.

Proof. Due to the existence of z_C we know that $\Lambda_0[\psi]$ is well-defined. Let us first prove that for all $z \in CH(\psi; U)$ we have

(5.10)

$$\int_{\{\psi=0\}} \operatorname{div} z\, dx = \int_{\partial\{\psi\le 0\}} \sigma(\nu)\, d\mathcal{H}^{n-1} - \int_{\partial\{\psi\ge 0\}} \sigma(-\nu)\, d\mathcal{H}^{n-1} = C|\{\psi = 0\}|.$$

The characterization of $\partial\sigma$ in (5.4) yields $z\cdot\nabla\psi = \sigma(\nabla\psi)$ a.e. on $\{0 < |\psi| < \delta_0\}$. For $\varepsilon > 0$ let η_ε be the standard mollifier with radius ε and let $z_\varepsilon := z * \eta_\varepsilon$, where we extend z by 0 outside U. We have

$$\begin{aligned} z_\varepsilon \cdot \nabla\psi &\to \sigma(\nabla\psi) && \text{a.e. in } U, \\ \operatorname{div} z_\varepsilon &\to \operatorname{div} z && \text{in } L^2(U), \end{aligned}$$

as $\varepsilon \to 0$. The divergence theorem gives

$$\int_{\{|\psi|<\delta\}} \operatorname{div} z_\varepsilon\, dx = \int_{\partial\{|\psi|<\delta\}} z_\varepsilon \cdot \nu\, d\mathcal{H}^{n-1}.$$

By the coarea formula, $\nu = \frac{\nabla\psi}{|\nabla\psi|}$ $\mathcal{H}^{n-1}$-a.e. on $\partial\{\psi < \delta\}$ and $\nu = -\frac{\nabla\psi}{|\nabla\psi|}$ $\mathcal{H}^{n-1}$-a.e. on $\partial\{\psi > -\delta\}$ for a.e. $\delta \in (0, \delta_0)$. After sending $\varepsilon \to 0$, the dominated convergence theorem yields for a.e. $\delta \in (0, \delta_0)$

$$\int_{\{|\psi|<\delta\}} \operatorname{div} z \, dx = \int_{\partial\{\psi<\delta\}} \sigma(\nu) \, d\mathcal{H}^{n-1} - \int_{\partial\{\psi>-\delta\}} \sigma(-\nu) \, d\mathcal{H}^{n-1}.$$

Sending $\delta \to 0$ along a sequence leads to (5.10). We recover the second equality in (5.10) by recalling that $z_C \in CH(\psi; U)$ satisfies $\operatorname{div} z_C = C$ a.e. on $\{\psi = 0\}$.

Let us write $A = \{\psi = 0\}$. For $v = \operatorname{div} z_{\min}$ we have $\int_A v \, dx = \int_A C \, dx$ by (5.10) and therefore

$$\int_A v^2 \, dx = \int_A C^2 \, dx + \int_A (v - C)^2 \, dx \geq \int_A C^2 \, dx. \tag{5.11}$$

By patching Corollary 5.5, the vector field

$$\tilde{z} = z_C \mathbf{1}_A + z_{\min} \mathbf{1}_{U \setminus A}$$

is also a Cahn-Hoffman vector field with

$$\operatorname{div} \tilde{z} = \operatorname{div} z_C \mathbf{1}_A + \operatorname{div} z_{\min} \mathbf{1}_{U \setminus A} \qquad \text{a.e. in } U.$$

Therefore (5.11) implies that $\|\operatorname{div} z_{\min}\|_{L^2(U)} \geq \|\operatorname{div} \tilde{z}\|_{L^2(U)}$ and we conclude that $\operatorname{div} \tilde{z}$ is minimizing. By uniqueness, $\operatorname{div} z_{\min} = C$ a.e. on A. □

Let us conclude with a few examples of simple useful facets for which we can compute Λ_0 explicitly.

Example 5.12. Wulff facet.

For $R > 0$ consider $\psi(x) := \max(\sigma^\circ(x) - R, 0)$. We have $\{\psi = 0\} = RW_\sigma$.

Take U to be a sufficiently large open ball containing RW_σ and consider the vector field

$$z(x) := \begin{cases} \frac{x}{R}, & \sigma^\circ(x) \leq R, \\ \frac{x}{\sigma^\circ(x)}, & \text{otherwise.} \end{cases}$$

Clearly $z \in L^\infty(U)$ and $\operatorname{div} z \in L^2(U)$ with

$$\operatorname{div} z = \begin{cases} \frac{n}{R}, & \sigma^\circ(x) \leq R, \\ \frac{n-1}{\sigma^\circ(x)}, & \text{otherwise.} \end{cases}$$

It is easy to check that $z \in \partial\sigma(\nabla\psi)$ a.e. Therefore $CH(\psi; U) \neq \emptyset$ and $\Lambda_0[\psi] = \frac{n}{R}$ on $\{\psi = 0\}$ by Lemma 5.11.

Example 5.13. Wulff facet with a "concentric" hole; Fig. 10. This example is useful in calculating the crystalline curvature of the exterior facets of the sponge-like solution in Figure 8.

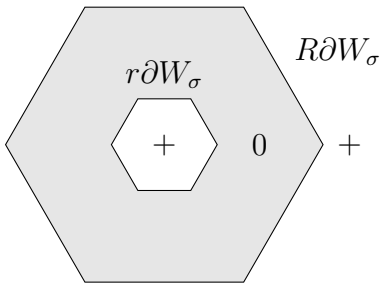

FIGURE 10. Wulff facet with a hole in Example 5.13, with sign of ψ indicated.

Suppose that σ° is even, i.e., $\sigma^\circ(-x) = \sigma^\circ(x)$ for all x. Consider $0 < r < R$ and the function

$$\psi(x) := \max(r - \sigma^\circ(x), 0, \sigma^\circ(x) - R),$$

so that $\{\psi = 0\} = RW_\sigma \setminus \operatorname{int} rW_\sigma$. Let us set

$$a := \frac{R^{n-1}r^{n-1}(R+r)}{R^n - r^n}, \qquad b := \frac{R^{n-1} + r^{n-1}}{R^n - r^n}$$

We claim that the vector field

$$z(x) := \begin{cases} -\frac{x}{\sigma^\circ(x)}, & 0 < \sigma^\circ(x) \le r, \\ \big(-a(\sigma^\circ(x))^{-n} + b\big)x, & r < \sigma^\circ(x) < R, \\ \frac{x}{\sigma^\circ(x)}, & \sigma^\circ(x) \ge R, \end{cases}$$

is a Cahn–Hoffman vector field for ψ on any U away from $x = 0$. To see that, we consider $g(s) := (-as^{-n} + b)s$. We note that $g(r) = -1$ and $g(R) = 1$, and g is increasing on $s > 0$ which yields $-1 < g(s) < 1$ for $r < s < R$. By the assumption that σ° is even, we have

$$\sigma^\circ(z(x)) = |g(\sigma^\circ(x))| \le 1 \qquad r < \sigma^\circ(x) < R.$$

This, by the characterization of the subdifferential, for example (5.4), implies that $z(x) \in \partial\sigma(0) = \partial\sigma(\nabla\psi(x))$ for $r < \sigma^\circ(x) < R$. For other x the inclusion $z(x) \in \partial\sigma(\nabla\psi(x))$ a.e. is obvious.

We also see that z is in fact locally Lipschitz continuous away from $x = 0$. Therefore $\operatorname{div} z \in L^2(U)$ for any U away from $x = 0$ and hence $z \in CH(\psi; U)$.

A direct computation using $x \cdot \nabla\sigma^\circ(x) = \sigma^\circ(x)$ yields that almost everywhere

$$\operatorname{div} z = \begin{cases} -\frac{n-1}{\sigma^\circ(x)}, & 0 < \sigma^\circ(x) \le r, \\ nb, & r < \sigma^\circ(x) < R, \\ \frac{n-1}{\sigma^\circ(x)}, & \sigma^\circ(x) \ge R. \end{cases}$$

In particular, $\Lambda_0[\psi] = nb = n\frac{R^{n-1}+r^{n-1}}{R^n-r^n}$ by Lemma 5.11, matching the formula (5.9).

Note that in the limit $r \to 0+$, z converges to the vector field in Example 5.12.

Example 5.14. Convex-concave facet; Fig. 10 with negative sign in the hole.

Consider $0 < r < R$ and the function

$$\psi(x) := \min(\sigma^\circ(x) - r, \max(0, \sigma^\circ(x) - R)),$$

so that again $\{\psi = 0\} = RW_\sigma \setminus \operatorname{int} rW_\sigma$, but this time $\psi < 0$ in rW_σ. The vector field

$$z(x) = \frac{x}{\sigma^\circ(x)}$$

is a Cahn–Hoffman vector field for ψ on any U away from 0.

But we can be more precise as in Example 5.13. Let us set

$$a := \frac{R^{n-1}r^{n-1}(R-r)}{R^n - r^n}, \qquad b := \frac{R^{n-1} - r^{n-1}}{R^n - r^n}$$

and consider the vector field

$$z(x) := \begin{cases} \frac{x}{\sigma^\circ(x)}, & 0 < \sigma^\circ(x) \le r, \\ \big(a(\sigma^\circ(x))^{-n} + b\big)x, & r < \sigma^\circ(x) < R, \\ \frac{x}{\sigma^\circ(x)}, & \sigma^\circ(x) \ge R. \end{cases}$$

This is a locally Lipschitz continuous vector field away from $x = 0$. Moreover, $\operatorname{div} z = nb$ almost everywhere for $r < \sigma^\circ(x) < R$.

Let us check that it is a Cahn–Hoffman vector field. The inclusion $z(x) \in \partial\sigma(\nabla\psi(x))$ is clear for $\sigma^\circ(x) < r$ and $R < \sigma^\circ(x)$. Since $\psi(x) \equiv 0$ for $r < \sigma^\circ(x) < R$, we only need to check that $\sigma^\circ(z) \le 1$ by (5.4).

The function $g(s) := (as^{-n} + b)s$ is convex on $s > 0$ with minimum at $\hat{s} = (\frac{b}{(n-1)a})^{-1/n}$ with value $g(\hat{s}) = \frac{n}{n-1}b\hat{s} > 0$. Therefore $as^{-n} + b > 0$ for $s > 0$ and we have

$$\sigma^\circ(z(x)) = g(\sigma^\circ(x)) \qquad r < \sigma^\circ(x) < R.$$

Since $g(r) = g(R) = 1$, by convexity of g we conclude that $\sigma^\circ(z(x)) \le 1$ for all $x \ne 0$. Therefore z is a Cahn–Hoffman vector field with constant divergence on the facet $\{\psi = 0\}$ and hence by Lemma 5.11 we have $\Lambda_0[\psi] = nb$.

5.4. **Comparison and approximation.** We start with the comparison principle for the σ°-(L^2) divergence. Here $\operatorname{sign} s = -1, 0, 1$ if $s < 0$, $s = 0$, or $s > 0$ respectively.

Theorem 5.15. *Let σ be a convex, positively one-homogeneous function on $\mathbb{R}^n$ that is positive away from 0. Let U be an open subset of $\mathbb{R}^n$ and let $\psi_i \in Lip(U)$ with $\{\psi_i = 0\}$ compact subset of U, for $i = 1, 2$. Let $f_i \in L^2(U)$ for $i = 1, 2$. If $CH(\psi_i; U) \neq \emptyset$ for $i = 1, 2$ and*

$$\operatorname{sign} \psi_1 \leq \operatorname{sign} \psi_2, \qquad f_1 \geq f_2$$

then

$$\Lambda_{f_1}[\psi_1] \leq \Lambda_{f_2}[\psi_2] \qquad \text{a.e. on } \{\psi_1 = 0\} \cap \{\psi_2 = 0\}.$$

Before we proceed with the proof, we recall here a technical lemma proved in [91, Lemma 4.13]. It is a variant of a result for continuous functions established in [45, 60]; see also [83, Lemma 4.2.9].

Lemma 5.16. *Suppose that ψ and φ are two nonnegative periodic Lipschitz functions on $\mathbb{R}^n$, such that $\{\psi = 0\} \subset \{\varphi = 0\}$. Then there exists a Lipschitz continuous function $\theta : [0, \infty) \to [0, \infty)$ such that $\theta(0) = 0$, $\theta(s) > 0$ for $s > 0$ and $\theta'(s) > 0$ for almost every $s > 0$ and we have*

$$\theta \circ \varphi \leq \psi \qquad \text{on } \mathbb{R}^n.$$

Proof of Theorem 5.15. For simplicity, we assume $f_1 = f_2 = 0$. We can assume that U is connected and by scaling that $U \subset (-\frac{1}{2}, \frac{1}{2})$. By making U smaller if necessary, we may assume that $\min_{\partial U} |\psi_i| > 0$.

We can modify ψ_i away from $\{\psi_i = 0\}$ to make it nonzero constant near ∂U and then extend it using this constant periodically so that $\psi_i \in Lip(\mathbb{T}^n)$ and $CH(\psi_i; \mathbb{T}^n) \neq \emptyset$ without changing the value of $\Lambda_0[\psi_i]$ on $\{\psi_i = 0\}$. If $n = 1$ we might have to do an even extension first if the sign differs on $\inf U$ and $\sup U$.

By Lemma 5.16, we can also find $\theta_1, \theta_2 \in Lip(\mathbb{R})$ with $\theta_i(0) = 0$ and $\theta_i' > 0$ a.e. so that $\theta_1 \circ \psi_1 \leq \theta_2 \circ \psi_2$ everywhere. Since by Remark 5.10 we have $CH(\theta_i \circ \psi_i; \mathbb{T}^n) = CH(\psi_i; \mathbb{T}^n)$, we can replace ψ_i with $\theta_i \circ \psi_i$ and assume that $\psi_1 \leq \psi_2$ on $\mathbb{T}^n$.

We then have $\Lambda_0[\psi_i] = -\partial^0 \mathcal{E}(\psi_i)$ on $\{\psi_i = 0\}$ by the characterization of the subdifferential in Corollary 5.2. Due to the comparison Proposition 5.7, we have $\psi_1 \leq \psi_2$ implies $\psi_{1,a} \leq \psi_{2,a}$ for the solutions of the resolvent problem (5.7). On $\{\psi_1 = 0\} \cap \{\psi_2 = 0\}$ we have

$$\frac{\psi_{1,a} - \psi_1}{a} \leq \frac{\psi_{2,a} - \psi_2}{a},$$

which using the convergence in Proposition 5.6 and sending $a \to 0$ implies $-\partial^0 \mathcal{E}(\psi_1) \leq -\partial^0 \mathcal{E}(\psi_2)$ a.e. on $\{\psi_1 = 0\} \cap \{\psi_2 = 0\}$. □

Abstract facets. The comparison principle for $\Lambda_f[\psi]$ implies that the value on $\{\psi = 0\}$ depends only on f and $\operatorname{sign}\psi$. We define the relation $\sim$ on $\mathcal{F} = \{\xi \mid \xi : \mathbb{R}^n \to \mathbb{R}\}$ the set of all real-valued function on $\mathbb{R}^n$ as

$$\xi_1 \sim \xi_2 \qquad \Leftrightarrow \qquad \operatorname{sign}\xi_1 = \operatorname{sign}\xi_2.$$

This relation is an equivalence relation on $\mathcal{F}$. We refer to its equivalence classes $[\xi] := \{\psi \mid \psi \sim \xi\} \subset \mathcal{F}$ as *(abstract) facets*. We write $[\xi_1] \preceq [\xi_2]$ when $\operatorname{sign}\xi_1 \leq \operatorname{sign}\xi_2$ and this relation defines a partial order on the set of all facets $\mathcal{F}/\sim := \{[\xi] \mid \xi \in \mathcal{F}\}$.

Cahn–Hoffman facet. We say that a facet $[\xi]$ is a σ°-(L^2) *Cahn–Hoffman facet* if $\{\xi = 0\}$ is compact and there are an open set $U \subset \mathbb{R}^n$, $\{\xi = 0\} \subset U$ and a Lipschitz function $\psi \in [\xi]$ such that $CH(\psi; U) \neq \emptyset$. The facets in Examples 5.12–5.14 are Cahn–Hoffman. It is not clear that this regularity condition can be compared with the notion of σ^0-regularity, or Lipschitz σ^0-regularity in the literature [24, 25, 27]

Proposition 5.17. *For σ°-(L^2) Cahn–Hoffman facets $[\chi_1]$ and $[\chi_2]$ and functions $f_i \in L^2(\{\chi_1 = 0\} \cup \{\chi_2 = 0\})$ we have*

$$[\chi_1] \preceq [\chi_2], \qquad f_1 \geq f_2 \text{ a.e.}$$

implies

$$\Lambda_{f_1}[\chi_1] \leq \Lambda_{f_2}[\chi_2] \text{ a.e. on } \{\chi_1 = 0\} \cap \{\chi_2 = 0\}.$$

We will use σ°-(L^2) Cahn-Hoffman facets to build test functions for viscosity solutions of the crystalline mean curvature flow and so we need to make sure there are enough of them. In fact, any facet with bounded zero set can be approximated by σ°-(L^2) Cahn-Hoffman facets monotonically arbitrarily close in the Hausdorff distance. The following theorem was proven in [81] for σ the Euclidean norm, and in [92] in full generality.

Theorem 5.18. *Let χ be an n-dimensional facet with $\{\chi = 0\}$ bounded and σ an anisotropy. Given $\rho > 0$ there exists a σ°-(L^2) Cahn-Hoffman facet $\tilde{\chi}$ such that $\chi(x) \leq \tilde{\chi}(x) \leq \sup_{|x-y|\leq\rho} \chi(y)$ for $x \in \mathbb{R}^n$.*

6. Approach by the theory of viscosity solutions

In this section we introduce a notion of *viscosity* solutions for nonlinear partial differential equations that include the very singular term $\operatorname{div}\nabla\sigma(\nabla u) - f$ that represents an anisotropic mean curvature with forcing.

For the definition of the anisotropic mean curvature we use the quantity Λ_f that was introduced in Section 5.3. It is important to note that if f depends on x, the term $\operatorname{div}\nabla\sigma(\nabla u) - f$ must be carefully defined together

and f cannot be added separately. Heuristically, the anisotropic mean curvature flow prefers flat facets in the singular directions of σ even in the presence of nonuniform forcing, and so the full quantity $\operatorname{div} \nabla\sigma(\nabla u) - f$ should be constant on facets. If we considered the forcing f separately in the definition of a viscosity solution, the comparison principle would still be valid however we would have a problem with stability in the approximation by regularized problems and ultimately we could not establish existence of solutions. For a counterexample to existence see [93, Sec. 6].

6.1. **Definition of viscosity solutions.** If $\sigma \in C^2(\mathbb{R}^n \setminus \{0\})$, it only has a singularity at $p = 0$ and we have everything we need to define the viscosity solution for (6.4). The following is the notion of the viscosity solution introduced in [81, 80] assuming that F does not depend on x and t and there is no forcing term.

Definition 6.1. *An upper semicontinuous function u on $\mathbb{R}^n \times (0, \infty)$ is a* viscosity subsolution *of*

$$u_t + F(\nabla u, \operatorname{div} \nabla\sigma(\nabla u)) = 0 \tag{6.1}$$

if the following two conditions hold:

(i) *(conventional test) If $\varphi \in C^2$ near $(\hat{x}, \hat{t})$, $\nabla\varphi(\hat{x}, \hat{t}) \neq 0$ and $u - \varphi$ has a local maximum at $(\hat{x}, \hat{t})$, then*

$$\varphi_t(\hat{x}, \hat{t}) + F(\nabla\varphi(\hat{x}, \hat{t}), \operatorname{div} \nabla\sigma(\nabla\varphi)(\hat{x}, \hat{t})) \leq 0. \tag{6.2}$$

(ii) *(faceted test) If $\varphi(x, t) = \psi(x) + g(t)$ with $g \in C^1(\mathbb{R})$ and $\psi \in Lip(\mathbb{R}^n)$ so that $[\psi]$ is a σ°-(L^2) Cahn-Hoffman facet, $\hat{x} \in \operatorname{int}\{\psi = 0\}$, $u - \varphi(\cdot - h)$ has a global maximum at $(\hat{x}, \hat{t})$ for all $|h|$ small, then there exists $\delta > 0$ such that*

$$g'(\hat{t}) + F(0, \operatorname*{ess\,inf}_{B_\delta(\hat{x})} \Lambda_0[\psi]) \leq 0. \tag{6.3}$$

A lower semi-continuous function is a viscosity supersolution *if it satisfies the above two conditions with maximum, $\leq$ and* $\operatorname{ess\,inf} \Lambda$ *replaced by minimum, $\geq$ and* $\operatorname{ess\,sup} \Lambda$*, respectively.*

Let us remark that in [81, 80] the facet test was restricted to test functions where the facet $[\psi]$ has a smooth boundary. However, this is not essential as was observed in later papers.

As you can see, we need to reduce the class of test functions testing at points where $\nabla u = 0$ to be even able to define a reasonable value of $\operatorname{div} \nabla\sigma(\nabla\varphi)$.

To include a forcing term f that depends on the x variable, we can follow [93] to modify the above definition. We introduce

$$\underline{\Lambda}_f[\xi](x) := \lim_{\delta\to 0+} \operatorname*{ess\,inf}_{B_\delta(x)} \Lambda_f[\xi], \qquad \overline{\Lambda}_f[\xi](x) := \lim_{\delta\to 0+} \operatorname*{ess\,sup}_{B_\delta(x)} \Lambda_f[\xi],$$

on the interior of $\{\xi = 0\}$, which are well-defined and finite by the comparison principle with Wulff facets in Example 5.12 as long as f is locally bounded. In fact, in this case $\underline{\Lambda}_f[\xi]$ is lower semi-continuous while $\overline{\Lambda}_f[\xi]$ is upper semi-continuous.

Then we can define a viscosity subsolution of the PDE

$$u_t + F(x, t, \nabla u, \operatorname{div} \nabla\sigma(\nabla u) - f) = 0 \tag{6.4}$$

following the above definition, but replacing (6.2) with

$$\varphi_t(\hat{x}, \hat{t}) + F(\hat{x}, \hat{t}, \nabla\varphi(\hat{x}, \hat{t}), \operatorname{div} \nabla\sigma(\nabla\varphi)(\hat{x}, \hat{t}) - f(\hat{x}, \hat{t})) \le 0.$$

and (6.3) with

$$g'(\hat{t}) + F(\hat{x}, \hat{t}, 0, \underline{\Lambda}_f[\psi](\hat{x})) \le 0. \tag{6.5}$$

This latter condition is slightly weaker than (6.3) used in [81, 80], and allows for the proof of stability to handle a non-constant driving force f.

If the anisotropy σ has singularities other than at $p = 0$, the faceted test has to be extended to those gradients of the solution. However, the singular set of σ might be in general very complicated and it is not clear how to define a viscosity solution for a general convex anisotropy σ (or a convex function σ) except in one dimension.

Therefore we restrict our attention to crystalline anisotropies: an anisotropy σ is called crystalline if it is a maximum of a finite number of linear functions. In this case, the structure of singularities of σ is relatively simple. The "kind" of singularity is determined by the dimension of the subdifferential $\partial\sigma(p)$, which corresponds to the expected dimension of the facet in the direction p. We introduce the following orthogonal decomposition of the space $\mathbb{R}^n$. For a fixed gradient $\hat{p} \in \mathbb{R}^n$, define Z to be the linear subspace of $\mathbb{R}^n$ parallel to the affine hull of $\partial\sigma(\hat{p})$, see Figure 11. In other words, Z is the smallest linear subspace such that $\partial\sigma(\hat{p}) \subset Z + \xi$ for some $\xi \in \mathbb{R}^n$. Set $k := \dim \partial\sigma(\hat{p}) := \dim Z$. We have an orthogonal decomposition $\mathbb{R}^n = Z \oplus Z^\perp$. We fix orthonormal bases of Z, $Z^\perp$ which give two linear isometries $\mathcal{T} : \mathbb{R}^k \to Z$ and $\mathcal{T}_\perp : \mathbb{R}^{n-k} \to Z^\perp$. This allows us to write any $x \in \mathbb{R}^n$ uniquely as $x = \mathcal{T}x' + \mathcal{T}_\perp x''$ for some $x' \in \mathbb{R}^k$ and $x'' \in \mathbb{R}^{n-k}$. For $k = 0$, $k = n$ we take $x = x''$ and $x = x'$, respectively. If we denote the adjoint of $\mathcal{T}$ as $\mathcal{T}^*$ and of $\mathcal{T}_\perp$ as $\mathcal{T}_\perp^*$, we have $x' = \mathcal{T}^* x$ and $x'' = \mathcal{T}_\perp^* x$.

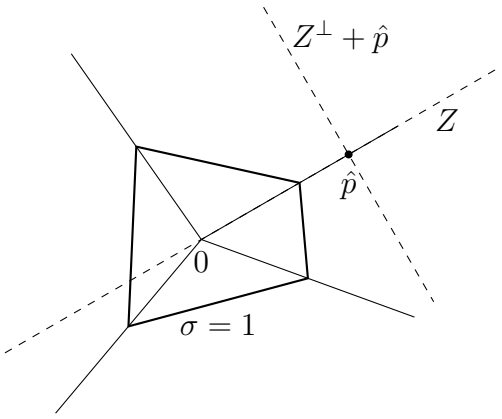

FIGURE 11. An illustration of the orthogonal decomposition $\mathbb{R}^2 = Z \oplus Z^\perp$ at $\hat{p}$ that lies on a "one-dimensional" singularity of σ. The thick polygon is the boundary $\partial F_\sigma = \{\sigma = 1\}$, and the solid rays from the origin indicate where $\dim \partial\sigma = 1$.

Using the above decomposition, we can "slice" the anisotropy σ to extract only the part that contains the singularity by introducing

$$\sigma^{\mathrm{sl}}_{\hat{p}}(w) := \lim_{\lambda \to 0+} \frac{\sigma(\hat{p} + \lambda \mathcal{T} w) - \sigma(\hat{p})}{\lambda}, \qquad w \in \mathbb{R}^k.$$

This sliced function is again positively one-homogeneous and so we can introduce a curvature-like quantity $\Lambda_{\hat{p},f}[\psi]$ for $(\sigma^{\mathrm{sl}}_{\hat{p}})^\circ$-$(L^2)$ Cahn–Hoffman facets $[\psi]$ on $\mathbb{R}^k$ and $f \in \mathbb{R}^k$.

Let us give the definition of viscosity solution assuming that $f \equiv 0$ that appeared in [92, Def. 4.7] with F independent of x and t and σ purely crystalline.

Definition 6.2. *An upper semicontinuous function u on $\mathbb{R}^n \times (0, \infty)$ is a* viscosity subsolution *of*

$$u_t + F(\nabla u, \operatorname{div} \nabla \sigma(\nabla u)) = 0 \tag{6.6}$$

if whenever $\hat{p} \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^n$, $\hat{t} \in (0, T)$ and φ is a stratified *test function $\varphi(x,t) = \psi(x') + \theta(x'') + \hat{p} \cdot x + g(t)$ with $g \in C^1(\mathbb{R})$, $\theta \in C^1(\mathbb{R}^{n-k})$ satisfying $\nabla\theta(\hat{x}'') = 0$, and $\psi \in Lip(\mathbb{R}^k)$ so that $[\psi]$ is a $(\sigma^{\mathrm{sl}}_{\hat{p}})^\circ$-$(L^2)$ Cahn-Hoffman facet, $\hat{x}' \in \operatorname{int}\{\psi = 0\}$, and $u - \varphi(\cdot - h)$ has a global maximum at $(\hat{x}, \hat{t})$ for all $h = \mathcal{T}h'$ with $|h'|$ small, then*

$$g'(\hat{t}) + F(\hat{p}, \operatorname*{ess\,inf}_{B_\delta(\hat{x})} \Lambda_{\hat{p},0}[\psi]) \le 0. \tag{6.7}$$

A lower semi-continuous function is a viscosity supersolution *if it satisfies the above two conditions with maximum, $\le$ and* ess inf *replaced by minimum, $\ge$ and* ess sup*, respectively.*

When a forcing f that depends on the x variable is involved, the condition (6.7) has to be weakened as in (6.5), replacing $\operatorname{ess\,inf} \Lambda_{\hat{p},0}$ by $\underline{\Lambda}_{\hat{p},\hat{f}}$, $\hat{f}(w) = f(\hat{x}+\mathcal{T}w)$, for the stability with respect to an approximation by regularized problems to hold. See [93, Def. 2.7] for more details.

6.2. **Comparison principle.** In this section we review the comparison principle for the PDE (6.4). There are a few different versions available depending on the assumptions on F, σ and f.

Let us first suppose that $f \equiv 0$. The comparison theorem was first proved in [81, 80] in the setting of an anisotropic total variation flow with smooth anisotropy $\sigma \in C^2(\mathbb{R}^n \setminus \{0\})$ with σ^2 strictly convex and F independent of the x and t variables in the sense of Definition 6.1 on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We follow [80, Th. 4.1].

Theorem 6.3. *Let $\sigma \in C^2(\mathbb{R}^n \setminus \{0\}$ be an anisotropy and $F \in C(\mathbb{R}^n \times \mathbb{R})$ be nonincreasing in the second variable. Let u and v be respectively a bounded viscosity subsolution and a viscosity supersolution of (6.6) in the sense of Definition 6.1 on $\mathbb{T}^n \times [0,T]$. If $u \le v$ at $t = 0$ then $u \le v$ on $\mathbb{T}^n \times (0,T)$.*

When σ is crystalline while there is still no forcing, $f \equiv 0$, and F does not depend on x and t, the following comparison principle for viscosity solutions in the sense of Definition 6.2 was proved in [91, 92]. We follow the statement in [92, Th. 1.4]. Instead of a torus, the solutions are assumed to be constant outside of a ball.

Theorem 6.4. *Let σ be a crystalline anisotropy and $F \in C(\mathbb{R}^n \times \mathbb{R})$ be nonincreasing in the second variable, $F(0,0) = 0$. Suppose that u is a viscosity subsolution and v is a viscosity supersolution of (6.6) on $\mathbb{R}^n \times (0,T)$ in the sense of Definition 6.2 and that there are constants $R > 0$, $a \le b$ such that $u = a$ and $v = b$ on $(\mathbb{R}^n \setminus B_R(0)) \times (0,T)$. Then if $u \le v$ at $t = 0$, then $u \le v$ on $\mathbb{R}^n \times (0,T)$.*

If we consider a forcing term f that depends on x and t and solutions of (6.4), an analogous comparison principle was proven in [93, Th. 3.1]. However, in this case at least one of u or v must be continuous, and either it is also Lipschitz, or further regularity of F must be assumed. For details see [93].

For simplicity of exposition, we present here the proof of a comparison theorem in a simplified setting of a stationary problem. We consider the

stationary equation with $\sigma(p) = |p|$, $\nabla\sigma(p) = \frac{p}{|p|}$, that reads

$$u - \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f, \tag{6.8}$$

where $f \in C(\mathbb{R}^n)$ is given. This equation is of elliptic type. The definition of viscosity solution is naturally modified to the following.

An upper semicontinuous function u is a viscosity subsolution of (6.8) if

- If $\varphi \in C^2$ near $\hat{x}$, $\nabla\varphi(\hat{x}) \neq 0$ and $u - \varphi$ has a local maximum at $\hat{x}$, then
$$u(\hat{x}) - \operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)(\hat{x}) \leq f(\hat{x}).$$
- If $\varphi \in Lip$ so that $[\varphi]$ is a σ°-(L^2) Cahn-Hoffman facet, $\hat{x} \in \operatorname{int}\{\varphi = 0\}$, $u - \varphi(\cdot - h)$ has a global maximum at $\hat{x}$ for all $|h|$ small, then
$$u(\hat{x}) - \underline{\Lambda}_f[\varphi](\hat{x}) \leq 0.$$

A definition of a supersolution and a solution can be modified analogously.

Recall that

$$\operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) = \frac{1}{|\nabla\varphi|}\operatorname{tr}\left[\left(I - \frac{\nabla\varphi\otimes\nabla\varphi}{|\nabla\varphi|^2}\right)\nabla^2\varphi\right].$$

Remark 6.5. Note that the proof in this section applies with a small modification to general problems

$$F\left(u, \nabla u, \operatorname{div}\left(\tfrac{\nabla u}{|\nabla u|}\right)\right) = 0,$$

where $F : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R} \to \mathbb{R}$ is a continuous function, satisfying monotonicity

$$\begin{aligned} &F(r,p,\xi) \leq F(r,p,\eta) \qquad \text{for } \xi \geq \eta,\\ &F(r,p,\xi) \leq F(s,p,\xi) - \mu(s-r) \qquad \text{for } r < s, \end{aligned}$$

where $\mu > 0$ is a constant. We write the proof for $F(r,p,\xi) := r - \xi$.

Theorem 6.6. *Suppose that u and v are a viscosity subsolution and a viscosity supersolution on $\mathbb{R}^n$, respectively. Furthermore, assume that u and v are bounded. If there exist constants $R > 0$ and $a \leq b$ such that $u \equiv a$ and $v \equiv b$ on $\mathbb{R}^n \setminus B_R(0)$ then $u \leq v$ everywhere.*

To show this theorem, we assume that the conclusion does not hold and

$$m_0 := \sup(u - v) = \max(u - v) > 0.$$

To reach a contradiction, we double variables and for $\zeta \in \mathbb{R}^n$ we consider the function

$$\Phi_\zeta(x,y) = u(x) - v(y) - \frac{|x - y - \zeta|^2}{2\varepsilon}.$$

We consider the maximum of Φ_ζ as a function of ζ, that is,

$$\ell(\zeta) = \sup_{x,y} \Phi_\zeta.$$

It is convenient to introduce the set of points of maxima

$$\mathcal{A}(\zeta) := \arg\max \Phi_\zeta := \{(x,y) \mid \Phi_\zeta(x,y) = \ell(\zeta)\}$$

and the set of gradients of $\frac{|x-y-\zeta|^2}{2\varepsilon}$ at these points

$$\mathcal{B}(\zeta) := \left\{ \frac{x-y-\zeta}{\varepsilon} \mid (x,y) \in \mathcal{A}(\zeta) \right\}.$$

The parameter $\varepsilon > 0$ determines how much we penalize $x \neq y$. We have the following standard estimate on $|x-y|$; see [70]. We give a proof for completeness.

Lemma 6.7. *There is $C > 0$ such that for all $\varepsilon > 0$, $|\zeta| \leq \sqrt{m_0 \varepsilon}$ we have*

$$|x-y| \leq C\sqrt{\varepsilon} \qquad \text{for all } (x,y) \in \mathcal{A}(\zeta). \tag{6.9}$$

Furthermore

$$\ell(\zeta) \geq \frac{m_0}{2}.$$

Proof. First note that

$$\ell(\zeta) \geq \sup_x (u(x) - v(x)) - \frac{|\zeta|^2}{2\varepsilon} \geq m_0 - \frac{m_0}{2} = \frac{m_0}{2}.$$

Let M be a bound on $u(x) - v(y)$. Then for $(x,y) \in \mathcal{A}(\zeta)$ we obtain

$$M - \frac{|x-y-\zeta|^2}{2\varepsilon} \geq u(x) - v(y) - \frac{|x-y-\zeta|^2}{2\varepsilon} = \ell(\zeta) \geq \frac{m_0}{2} > 0.$$

Hence

$$|x-y| \leq \sqrt{2M\varepsilon} + |\zeta| \leq \sqrt{2M\varepsilon} + \sqrt{m_0\varepsilon}.$$

□

Let ω_f be the modulus of continuity of f, that is, $\omega : [0,\infty) \to [0,\infty)$, $\omega(0) = 0$, ω nondecreasing such that $|f(x) - f(y)| \leq \omega_f(|x-y|)$ for all x, y. Let $\varepsilon_0 > 0$ be such that $\omega_f(C\sqrt{\varepsilon_0}) < \frac{m_0}{4}$, where C is from (6.9). We consider two possible cases:

1. There exist $0 < \varepsilon < \varepsilon_0$ and $|\zeta| \leq \sqrt{m_0\varepsilon}$ such that $\mathcal{B}(\zeta) \neq \{0\}$.
2. $\mathcal{B}(\zeta) = \{0\}$ for all $|\zeta| \leq \sqrt{m_0\varepsilon}$, $0 < \varepsilon < \varepsilon_0$.

Case 1. We can fix ε, ζ and $(x,y) \in \mathcal{A}(\zeta)$ such that $0 < \varepsilon < \varepsilon_0$, $|\zeta| \leq \sqrt{m_0\varepsilon}$ and $p := \frac{x-y-\zeta}{\varepsilon} \neq 0$. Note that this choice together with Lemma 6.7 implies

$$|f(x) - f(y)| \leq \omega_f(|x-y|) \leq \omega_f(C\sqrt{\varepsilon}) \leq \omega_f(C\sqrt{\varepsilon_0}) \leq \frac{m_0}{4}, \tag{6.10}$$

and

$$u(x) - v(y) \geq u(x) - v(y) - \frac{|x-y-\zeta|^2}{2\varepsilon} = \ell(\zeta) \geq \frac{m_0}{2}.$$

Since the operator is smooth near $\nabla u \neq 0$, we are well within the classical viscosity solution framework for continuous operators. In particular, we can use the now standard maximum principle for semicontinuous functions, see the exposition in [47] or [83]. Therefore there exist symmetric matrices X, Y with $X \leq Y$, sequences $x_n \to x$, $y_n \to y$ and sequences of C^2 functions φ_n, ψ_n such that $u - \varphi_n$ has a local maximum at x_n, $v - \psi_n$ has a local minimum at y_n, and

$$(x_n, u(x_n), \nabla\varphi_n(x_n), \nabla^2\varphi_n(x_n)) \to (x, u(x), p, X),$$
$$(y_n, v(y_n), \nabla\psi_n(y_n), \nabla^2\psi_n(y_n)) \to (y, u(y), p, Y).$$

From the definition of viscosity solution we deduce

$$u - \frac{1}{|\nabla\varphi_n|} \operatorname{tr}\left[\left(I - \frac{\nabla\varphi_n \otimes \nabla\varphi_n}{|\nabla\varphi_n|^2}\right)\nabla^2\varphi_n\right] \leq f \qquad \text{at } x_n$$
$$v - \frac{1}{|\nabla\psi_n|} \operatorname{tr}\left[\left(I - \frac{\nabla\psi_n \otimes \nabla\psi_n}{|\nabla\psi_n|^2}\right)\nabla^2\psi_n\right] \geq f \qquad \text{at } y_n.$$

In the limit $n \to \infty$, continuity and (6.10) yield

$$u(x) - \frac{1}{|p|} \operatorname{tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right] \leq f(x)$$
$$\leq f(y) + \frac{m_0}{4} \leq v(y) - \frac{1}{|p|} \operatorname{tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) Y\right] + \frac{m_0}{4}.$$

On the other hand, $u(x) \geq v(y) + \frac{m_0}{2}$ and $X \leq Y$ imply

$$v(y) - \frac{1}{|p|} \operatorname{tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) Y\right] + \frac{m_0}{4} < u(x) - \frac{1}{|p|} \operatorname{tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right].$$

We reach a contradiction.

Case 2. Since we cannot find any maximum of Φ_ζ at which the "gradient" of u nor v is nonzero, we need to construct admissible faceted test functions for the faceted test in the definition of viscosity solution.

The extra parameter $|\zeta|$ provides a little bit of space to construct these faceted test functions. The following "constancy" lemma was proven in a more general settings in [70, Lemma 7.5]. We include the proof in our simple setting for completeness.

Lemma 6.8 (Constancy)**.** *Let $G \subset \mathbb{R}^n$ be a closed ball. If for all $\zeta \in G$ there exists $(x, y) \in \mathcal{A}(\zeta)$ such that $x - y - \zeta = 0$ then $\ell(\zeta)$ is constant on G.*

Proof. Take $\zeta, \mu \in G$ and $(x, y) \in \mathcal{A}(\zeta)$ with $x - y - \zeta = 0$. In particular, $\ell(\zeta) = u(x) - v(y)$. From the definition of ℓ,

$$\ell(\mu) \geq u(x) - v(y) - \frac{|x - y - \mu|^2}{2\varepsilon} = \ell(\zeta) - \frac{|x - y - \mu|^2}{2\varepsilon}.$$

Since $x - y - \zeta = 0$, we have

$$|x - y - \mu|^2 = |x - y - \zeta + \zeta - \mu|^2 = |\zeta - \mu|^2,$$

yielding

$$\ell(\mu) - \ell(\zeta) \geq -\frac{|\zeta - \mu|^2}{2\varepsilon},$$

and, by symmetry,

$$|\ell(\mu) - \ell(\zeta)| \leq \frac{|\zeta - \mu|^2}{2\varepsilon} \qquad \text{for all } \zeta, \mu \in G.$$

We conclude that $\ell(\zeta) = \ell(\mu)$ for all $\zeta, \mu \in G$. □

Since in Case 2 we have $\mathcal{B}(\zeta) = \{0\}$ for all $\zeta \in G := \overline{B}_{\sqrt{m_0\varepsilon}}$, we can choose $(\hat{x}, \hat{y}) \in \mathcal{A}(0)$ with $\hat{x} - \hat{y} = 0$ and the above lemma yields

(6.11)
$$u(x) - v(y) \leq \ell(x - y) = \ell(0) = u(\hat{x}) - v(\hat{x}) \qquad \text{for } |x - y| \leq \sqrt{m_0\varepsilon}.$$

Let us set $\lambda = \sqrt{m_0\varepsilon}$. Defining

$$\eta_u = \operatorname{sign}(u - u(\hat{x})) \qquad \eta_v = \operatorname{sign}(v - v(\hat{x})),$$

the inequality (6.11) yields

$$\sup_{\overline{B}_{\lambda/2}(x)} \eta_u \leq \inf_{\overline{B}_{\lambda/2}(x)} \eta_v \qquad \text{for all } x.$$

By the density result Theorem 5.18, there are admissible facets χ_u and χ_v satisfying

$$\sup_{\overline{B}_{\lambda/4}(x)} \eta_u \leq \chi_u(x) \leq \sup_{\overline{B}_{\lambda/2}(x)} \eta_u \leq \inf_{\overline{B}_{\lambda/2}(x)} \eta_v(x) \leq \chi_v \leq \inf_{\overline{B}_{\lambda/4}(x)} \eta_v.$$

Clearly $\chi_u = \chi_v = 0$ on $B_{\lambda/4}(\hat{x})$. By the comparison principle for the curvature operator, Proposition 5.17,

$$\operatorname*{ess\,inf}_{B_{\lambda/4}(\hat{x})} \Lambda_f[\chi_u] \leq \operatorname*{ess\,sup}_{B_{\lambda/4}(\hat{x})} \Lambda_f[\chi_v],$$

which implies

$$\underline{\Lambda}_f[\chi_u](\hat{x}) \leq \overline{\Lambda}_f[\chi_v](\hat{x}). \tag{6.12}$$

Let us choose an admissible support function $\varphi_u \in Lip \cap [\chi_u]$. Since u is bounded and upper semicontinuous, we can multiply the positive part of φ_u

by a large positive constant, and the negative part of φ_u by a small positive constant, if necessary, to guarantee that

$$u \le \varphi_u(\cdot - h) + u(\hat{x}) \qquad \text{for } |h| < \tfrac{\lambda}{8}.$$

Note that the equality is attained at $\hat{x}$ as $\varphi_u = 0$ on $B_{\lambda/4}(\hat{x})$. Therefore φ_u is an admissible faceted test function for the viscosity solution test and

$$u(\hat{x}) - \underline{\Lambda}_f[\varphi_u](\hat{x}) \le 0.$$

Similarly, we can find $\varphi_v \in Lip \cap [\chi_v]$ with

$$v(\hat{x}) - \overline{\Lambda}_f[\varphi_v](\hat{x}) \ge 0.$$

Thus, recalling (6.12), we have

$$u(\hat{x}) \le \underline{\Lambda}_f[\varphi_u](\hat{x}) \le \overline{\Lambda}_f[\varphi_v] \le v(\hat{x}) \le u(\hat{x}) - m_0 < u(\hat{x}),$$

a contradiction. This finishes the proof of Theorem 6.6.

6.3. **Existence of solutions.** The existence of viscosity solutions is usually established using Perron's method: the largest subsolution of the problem is automatically a solution. However, the operation of taking a supremum of a class of viscosity solutions requires a stability property whose validity is unclear for the viscosity solutions considered here in dimensions $n \ge 2$. In one dimension, Perron's method was used to construct viscosity solutions for (6.4) in [79]. This however requires a careful treatment of the nonlocal anisotropic curvature.

The main issue with the stability required for the supremum of subsolutions to be a subsolution is the discontinuity of the value of $\Lambda_f[\psi]$ when a facet bends or breaks. In the standard proof of this stability, it is crucial to localize by replacing a test function φ by another so that $u - \varphi$ can be assumed to have a *strict* local maximum (or minimum). Due to the discontinuity of our operator Λ_f with respect to such bending, this tool is not available.

The approach that was taken in [81, 91] is via stability with respect to approximation by problems with regularized σ. In particular, we consider two ways of approximating a crystalline σ:

(a) $\sigma_m \in C^2(\mathbb{R}^n)$ with $a_m^{-1} I \le \nabla^2 \sigma_m \le a_m I$ for some $a_m > 0$, σ_m is a decreasing sequence with $\sigma_m \to \sigma$ locally uniformly.
(b) σ_m are anisotropies with $\sigma_m \in C^2(\mathbb{R}^n \setminus \{0\})$ such that σ_m^2 is strictly convex and $\sigma_m \to \sigma$ locally uniformly.

However, for various reasons related to the regularity of the solutions of the approximating problems, we need to assume that F does not depend on the x variable, and consider solutions of the regularized problems

$$u_t + F(t, \operatorname{div} \nabla\sigma_m(\nabla u) - f) = 0. \tag{6.13}$$

Since σ_m are C^2 and convex, the classical theory of viscosity solutions applies, including the unique existence of solutions for given bounded continuous initial data.

We have the following stability result when approximating using (a), see [93, Th. 4.1] or [91, Th. 8.1], which resembles the usual stability of viscosity solutions in the classical theory. Let $\limsup^*$ (resp. $\liminf_*$) denote the relaxed upper limit (lower) limit defined by

$$(\limsup{}^* u_m)(x,t) = \limsup_{m\to\infty} \left\{ u_k(y,s) \;\middle|\; |x-y| + |t-s| + 1/k < 1/m \right\}$$
$$(\liminf{}_* u_m)(x,t) = -\left(\limsup(-u_m)\right)(x,t)$$

for a sequence $\{u_m\}$ of functions on $\mathbf{R}^n \times [0,\infty)$.

Theorem 6.9. *Let σ be a crystalline anisotropy and assume that F does not depend on the x variable and $f \in C(\mathbb{R}^n \times \mathbb{R})$ is Lipschitz continuous in space, uniformly in time. If $\{u_m\}$ is a locally bounded sequence of viscosity solutions of (6.13) with σ_m as in (a) above, then $\limsup^*_{m\to\infty} u_m$ is a viscosity subsolution of (6.4), and $\liminf_{*m\to\infty} u_m$ is a viscosity supersolution of (6.4).*

The main idea of the proof of Theorem 6.9 is inspired by the perturbed test function method due to Evans [58]. Let us for simplicity assume that $f \equiv 0$. The crystalline mean curvature, or specifically the operator $\Lambda_0[\psi]$ is nonlocal on the facets of ψ. In contrast, the elliptic operators $\operatorname{div} \nabla\sigma_m(\nabla\psi)$ are local and they are in fact zero on the facets of ψ. To recover the nonlocal information in the limit $m \to \infty$, we *perturb* the test function ψ using a sequence ψ_m of uniformly converging C^2 functions $\psi_m \to \psi$, such that $\operatorname{div} \nabla\sigma_m(\nabla\psi_m)$ approximates the value of $\Lambda_0[\psi]$ in a suitable sense at the contact point.

Such approximation is available via the resolvent problem for the regularized energy $\mathcal{E}_m$, with σ replaced by σ_m in (5.2). For a given $a > 0$ and $\psi \in L^2(\mathbb{T}^n)$, there exists a unique solution $\psi_{a,m} \in L^2(\mathbb{T}^n)$ of

$$\psi_{a,m} + a\partial\mathcal{E}_m(\psi_{a,m}) \ni \psi.$$

If $\psi \in Lip(\mathbb{T}^n)$, then $\psi_{a,m}$ is Lipschitz uniformly in a and m by the comparison principle like Proposition 5.7 and translation invariance of the operator, and in fact it is C^2 by the elliptic regularity theory. Since $\mathcal{E}_m$ Mosco-converges to $\mathcal{E}$ (see (1.7) for the definition), we have a convergence of the

resolvent solutions $\psi_{a,m} \to \psi_a$ in $L^2(\mathbb{T}^n)$, see [12], and hence uniformly by the uniform Lipschitz continuity. Using Proposition 5.6, we can deduce that $\psi_{a,m}$ uniformly approximate ψ as $a \to 0$ and then $m \to \infty$. Functions $\psi_{a,m}$ are used to build test functions for the regularized problem, and allow us to deduce that $\limsup^*_{m\to\infty} u_m$ is a viscosity subsolution of (6.4), and $\liminf_{*m\to\infty} u_m$ is a viscosity supersolution of (6.4).

Approximation using (b) is relevant when considering the crystalline mean curvature flow as a limit of a smooth anisotropic mean curvature flow. To prove the stability for (b), we use the stability Theorem 6.9 to approximate each σ_m by a sequence of C^2 functions $\sigma_{m,\delta}$ and therefore we need to know that a given solution u_m can be approximated by a sequence of solutions $u_{m,\delta}$ with this anisotropy. This is known for example when u_m have continuous bounded initial data. We have the following stability result, [93, Th. 4.4].

Theorem 6.10. *Let σ, F and f be as in Theorem 6.9. Let $T > 0$ and let u_m be a locally bounded sequence of viscosity solutions of (6.13) on $\mathbb{R}^n \times (0,T)$ with σ_m as in (b) with initial data $u_m(\cdot,0) = u_{0,m}$, where $u_{0,m} \in C(\mathbb{R}^n)$ are uniformly bounded. Then $\limsup^*_{m\to\infty} u_m$ is a viscosity subsolution of (6.4), and $\liminf_{*m\to\infty} u_m$ is a viscosity supersolution of (6.4).*

Now with the stability with respect to approximation by the regularized problems established, and the comparison principle discussed in Section 6.2, we can follow the standard idea to show existence of (6.4) for given initial data when the operator F does not depend on the x variable. For given bounded uniformly continuous initial data, we take u_m solutions of the regularized problem with initial data u_0 from Theorem 6.10. By using barriers at $t = 0$, we can show that the limits satisfy

$$\liminf_{m\to\infty}{}_* u_m\big|_{t=0} \geq u_0, \qquad \limsup_{m\to\infty}{}^* u_m\big|_{t=0} \leq u_0.$$

From the comparison principle for (6.4) we immediately have

$$\limsup_{m\to\infty}{}^* u_m \leq \liminf_{m\to\infty}{}_* u_m.$$

This implies that both limits are equal, the convergence is locally uniform, and the limit is a viscosity solution of (6.4).

If the forcing f depends on x, there is an additional difficulty that the comparison principle for semi-continuous solutions is not available, see [93, Sec. 3]. The comparison principle established in [93] requires that at least one of the solutions is continuous. Fortunately, for operators F that come from the level set formulation of geometric motions one can prove uniform

Lipschitz bounds in space and uniform Hölder bounds in time on the approximating sequence u_m for Lipschitz initial data u_0, see [93, Sec. 5]. Therefore the convergence u_m is locally uniform for subsequences and the limits are a priori continuous. In particular, the restricted comparison principle applies and existence of solutions can be established. We have the following existence theorem, [93, Th. 1.1].

Theorem 6.11. *Assume that $g \in C(\mathcal{S}^{n-1} \times \mathbb{R})$ is Lipschitz continuous in the second variable uniformly in the first variable and non-decreasing in the second variable, σ is a crystalline anisotropy and $f \in C(\mathbb{R}^n \times \mathbb{R})$ is Lipschitz continuous in space uniformly in time. Then there is a unique global-in-time level set flow to*

$$V = g(\nu, \kappa_\sigma + f(x,t))$$

when the initial hypersurface is compact.

Remark 6.12. If f is constant, then the global Lipschitz continuity of F is unnecessary [91], [92]. In particular, it applies to (2.1). In the case $n = 2$, it applies to a general anisotropy under a slightly different definition of a solution [73]. Note that the level set equation for $V = \kappa_\sigma$ is

$$u_t = |\nabla u| \operatorname{div} \nabla\sigma(\nabla u)$$

so that each level set of u moves by $V = \kappa_\sigma$. The level set flow is a level set of a viscosity solution u. Its uniqueness (up to fattening) is guaranteed by the comparison principle and an invariance under a change of the dependent variable u (representing its level sets) together with Lemma 5.16. This procedure is standard for a level set flow; see e.g. [83]. The terminology of the level set flow here is different from that in Section 7.

6.4. **Convergence of various approximations.** It is well-known that the solution of the mean curvature flow equation is approximated by that of the Allen-Cahn equation; see [131], [34], [44], [59]. Anisotropic version of the Allen-Cahn equation is introduced by [129], which is an L^2-gradient flow of

$$F_\varepsilon(v) = \int_{\mathbb{R}^n} \left\{ \frac{1}{2}\sigma(\nabla v)^2 + \frac{1}{\varepsilon^2}\left(W(v) - \varepsilon\lambda F(v)\right) \right\} dx.$$

Here, $W(v)$ is a double-well potential typically $W(v) = (v^2-1)^2/2$ and $F(v) = Cv$ with constant C for simplicity. The parameter $\lambda > 0$ should be chosen in a suitable way. In an explicit form, the anisotropic Allen-Cahn equation reads

$$\beta(\nabla v) v_t - \operatorname{div}\left(\sigma(\nabla v)\zeta(\nabla v)\right) + \frac{1}{\varepsilon^2}\left(W'(v) - \varepsilon\lambda C\right) = 0 \tag{6.14}$$

with some kinetic coefficient $\beta > 0$ which is positively one-homogeneous; here $\zeta(p) = \nabla_p \sigma(p)$. If one considers an L^2 gradient flow of F, the equation (6.14) with $\beta \equiv 1$ is obtained. The extra multiplier comes from the modelling to include anisotropy of mobility. For a given closed interface Γ_0, we consider a function v_0^ε which converges in a suitable way to -1 in an open set surrounded by Γ_0 and to 1 outside the closure of the open set. It is expected that the solution of the anisotropic Allen-Cahn equation with initial data v_0^ε converges to 1 inside an open set surrounded by Γ_t and -1 outside Γ_t and this open set, where Γ_t is a (generalized) solution to the interface equation

$$\beta(\mathbf{n})V = \sigma(\mathbf{n})(\kappa_\sigma - C).$$

(Here λ should be taken as $\lambda = 2/3$ if $W(v) = (v^2-1)^2/2$.) Formal asymptotic analysis is carried out by [129], [166] and [29], which derives the interface equation. For smooth anisotropy with $\beta \equiv 1$, the convergence is established by [54] when the solution of the interface equation is smooth, here W is taken as double-obstacle type, for example, $W(v) = 1 - v^2$ in $|v| \le 1$ and $W(v) = \infty$ for $|v| > 1$. This result is extended when Γ_t is a generalized solution (a level-set solution allowing fattening). In [90] it is shown that such convergence is uniform in σ provided that the Frank diagram F_σ is bounded by a ball both from inside and outside. It does not depend on regularity of σ.

For crystalline σ under $\beta \equiv 1$, the convergence with some rate is established for planar crystalline flow [23]. It is somewhat extended to higher dimension for a special class of solutions of the interface equation; its existence is not clear [24]. Several explicit examples of convergence are given by [161]. One of the reasons why $\beta \equiv 1$ is assumed is that the notion of solutions for the Allen-Cahn equation is unclear. Maybe a viscosity approach will resolve this issue.

Since our solution for the interface equation for crystalline σ is obtained as a limit of smoother problems as in the previous subsection, combining uniform convergence with respect to σ we are able to prove the convergence as $\varepsilon \to 0$ by approximating β and σ by smooth function; see [90, Theorem 2.4]. Note that in two dimensional case, the stability was proved in [73].

Another typical way to approximate a solution is what is called Chambolle's scheme introduced by [39]. We here give its anisotropic version [38], [43]. We consider

$$V = M(\nu)\kappa_\sigma.$$

We set the support function of the polar of $1/M$ (Frank diagram of M) by M^0, i.e.,

$$M^0(x) := \sup\left\{x \cdot p \;\middle|\; |p| \le 1/M\,(p/|p|)\right\}.$$

Here M is assumed to be positive on S^{n-1}. The function M^0 is convex, positively 1-homogeneous in $\mathbb{R}^n$ and it is positive outside the origin. However, it may not satisfy the symmetry $M(x) = M(-x)$ so that $\mathrm{dist}_{M_0}(x,y) = M^0(x-y)$ is a non-symmetric distance. For a given bounded set E_0 in $\mathbb{R}^n$, let $d_{M^0}(x,E_0)$ denote its anisotropic signed distance, i.e.,

$$d_{M^0}(x,E_0) := \mathrm{dist}_{M^0}(x,E_0) - \mathrm{dist}_{M^0}(x,E_0^{\mathsf{c}}), \quad x \in \mathbb{R}^n,$$

where

$$\mathrm{dist}_{M^0}(x,E_0) := \inf_{y\in E_0} \mathrm{dist}_{M^0}(x,y).$$

We next consider an energy functional of the form

$$J_h(v,E_0) = \int_\Omega \left\{\sigma(\nabla v) + \frac{1}{2h}|v - d_{M^0}|^2\right\} dx$$

for a domain Ω containing E_0 with a small parameter $h > 0$. This value is finite in $L^2(\Omega) \cap BV(\Omega)$ so we regard J_h as a lower semicontinuous convex functional on $L^2(\Omega)$ by interpreting its value equal to ∞ on $L^2(\Omega)\backslash BV(\Omega)$. It admits a unique minimizer $w = \mathrm{argmin}\, J_h$. We introduce the operator T_h as

$$T_h(E_0) = \{x \in \mathbb{R}^n \mid w(x) \le 0\}.$$

An approximate flow is defined by applying the above step iteratively as

$$E^h(t) = T_h^{\lfloor t/h \rfloor}(E_0), \tag{6.15}$$

where $\lfloor s \rfloor$ denotes its integer part of $s > 0$. We expect that E^h converges to the level-set solution of $V = M(\nu)\kappa_\sigma$ as $h \to 0$, for example, in the Hausdorff distance sense uniformly in $t \in [0,T]$ with finite T. Let us give a very heuristic argument. We consider the isotropic case $V = \kappa$ so that $M = 1$ and $\sigma(p) = |p|$. Then the minimizer w satisfies the resolvent equation

$$\frac{w-d}{h} - \mathrm{div}\,\frac{\nabla w}{|\nabla w|} = 0,$$

where d denotes the Euclidean signed distance of E_0. This is the implicit Euler scheme for the total variation flow. The signed distance function satisfies $|\nabla d| = 1$ on the interface Γ_t so $V \approx \frac{w-d}{h}$ and it is expected that the zero level of w approximates the solution Γ_t.

The isotropic case of this scheme was first introduced in [39], which gives a monotone way to realize the time discrete scheme proposed by [2]; see also [128]. In [39] L^1 convergence: $E^h(t) \to E(t)$ on $[0,T]$, where $E(t)$ is the level set solution of $V = \kappa$ (starting from a closed set E_0 with $E_0 = \overline{\mathrm{int}\, E_0}$) was

established provided that no fattening phenomena occur. Its anisotropic extension is done by [38] in the case when E_0 is convex and compact under the assumption that σ/M is constant on S^{n-1}; see [43] for non-convex initial data; here anisotropy is assumed to be smooth. In [18] for a non-smooth σ including crystalline, a unique solution for $V = \sigma\kappa_\sigma$ is constructed when E_0 is convex and compact by defining a solution by the distance function. For smooth anisotropy for a bounded nonconvex initial data, the Hausdorff convergence is proved in [57], where they prove locally uniform convergence of an associated function

$$u^h(x,t) = \left(S_h^{[t/h]} u_0\right)(x)$$

with

$$(S^h u_0)(x) = \sup\left\{\mu \in \mathbb{R} \;\middle|\; x \in T_h\left(\{x \in \mathbb{R}^n \mid u_0(x) \geq \mu\}\right)\right\}.$$

Although it is remarked in [57] and [43], the case when σ and M are unrelated is not discussed in detail. In [106] a proof based on the distance function is given for several choices of σ and M and general initial data not necessarily compact mostly for smooth case. However, it is also shown in [106] that if the solution of crystalline anisotropy has a stability property we are able to prove the convergence of Chambolle's scheme by approximating M and σ. Since at that time, the stability was only available in two dimensional case [73], convergence result in [106] looks limited but it applies to general dimension at least for purely crystalline anisotropy since the stability holds for general dimension as discussed in the previous subsection. The reason why M and σ are approximated by a smoother one in Chambolle's scheme in [106] seems to avoid analysis for the resolvent equation for non-smooth M and σ, so it seems that it is not substantial.

In the next section we discuss a notion of solutions based on distance functions to the evolving surface that can be showed to be the limits of the discrete evolutions (6.15) given by Chambolle's scheme, see Theorem 7.7.

7. Approach by distance functions

In this section we discuss an alternative approach to defining a notion of solutions of the crystalline mean curvature flow that appeared in a series of papers by Chambolle, Morini, Novaga and Ponsiglione [42, 40, 41]. The main idea is to require that the distance function to an evolving set is a sub/supersolution of a related partial differential equation in the sense of distributions.

This approach applies to a form of the crystalline mean curvature flow that is *linear* in the curvature term:

$$V = M(\nu)(\kappa_\sigma - f). \tag{7.1}$$

However, both σ and M can be arbitrary anisotropies, not necessarily crystalline. For simplicity, we will assume that both σ and M are even, that is, $\sigma(p) = \sigma(-p)$ and $M(p) = M(-p)$ for all $p \in \mathbb{R}^n$. This restriction however does not appear to be essential. Moreover, the initial data E^0 can be an unbounded closed set, and the forcing term needs to be only $f \in L^\infty(\mathbb{R}^n \times (0,T))$ with $f(\cdot,t)$ Lipschitz uniformly in t.

The distance function must be adapted to the mobility M. As in [40] for any norm η we denote

$$\operatorname{dist}^\eta(x,E) := \inf_{y\in E} \eta(x-y), \qquad E \subset \mathbb{R}^n.$$

Note that $\operatorname{dist}^\eta(x,\emptyset) = +\infty$.

Let $E_n \subset \mathbb{R}^n$ be a sequence of closed sets and $E \subset \mathbb{R}^n$ a closed set. We say that E_n converges to E in *Kuratowski sense*, and write $E_n \overset{\mathcal{K}}{\to} E$, if $\operatorname{dist}^\eta(\cdot,E_n) \to \operatorname{dist}^\eta(\cdot,E)$ locally uniformly in $\mathbb{R}^n$ for some norm η. It is easy to see that if this converges for one norm, it converges for all norms.

The following definition appeared in [40].

Definition 7.1. *Let $E^0 \subseteq \mathbb{R}^n$ be a closed set. Let E be a closed set in $\mathbb{R}^n \times [0,+\infty)$ and for each $t \geq 0$ define $E(t) := \{x \in \mathbb{R}^n : (x,t) \in E\}$. We say that E is a* superflow *of (7.1) with initial datum E^0 if:*

(a) $E(0) \subseteq E^0$,

(b) $E(s) \overset{\mathcal{K}}{\to} E(t)$ *as* $s \nearrow t$ *for all* $t > 0$,

(c) *If* $E(t) = \emptyset$ *for some* $t \geq 0$, *then* $E(s) = \emptyset$ *for all* $s > t$.

(d) *Set* $T^* := \inf\{t > 0 : E(s) = \emptyset \text{ for } s \geq t\}$, *and*

$$d(x,t) := \operatorname{dist}^{M^\circ}(x, E(t)) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times (0,T^*) \setminus E.$$

Then there exists $K > 0$ such that the inequality

$$d_t \geq \operatorname{div} z + f - Kd \tag{7.2}$$

holds in the distributional sense in $\mathbb{R}^n \times (0,T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^n \times (0,T^*))$ such that $z \in \partial\sigma(\nabla d)$ a.e., $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^n \times (0,T^*) \setminus E$, and

$$(\operatorname{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^n \times (0,T^*) : d(x,t) \geq \delta\}) \qquad \text{for every } \delta \in (0,1).$$

An open set $A \subset \mathbb{R}^n \times [0,+\infty)$ is a subflow *of (7.1) with initial datum E^0 if A^{c} is a superflow of (7.1) with f replaced by $-f$ and with initial datum $(\operatorname{int} E^\circ)^{\mathsf{c}}$.*

A closed set $E \subset \mathbb{R}^n \times [0, +\infty)$ is a solution *of* (7.1) *with initial datum E^0 if it is a superflow and if* int E *is a subflow, both with initial datum E^0.*

The condition (b) is meant to prevent a possibility that E expands discontinuously, for example a bubble closing up, which cannot be ruled out by (7.2).

Note that K is related to the Lipschitz constant of f with respect to the distance induced by M. In fact, in the smooth case σ, M, $M^\circ \in C^2(\mathbb{R}^n \setminus \{0\})$, f continuous, then E is a superflow in the sense of Definition 7.1 if and only if $-\mathbf{1}_E$ is a viscosity supersolution of the level set equation

$$u_t = M(\nabla u)(\operatorname{div} \nabla \sigma(\nabla u) + f),$$

in $\mathbb{R}^n \times (0, T^*]$; see [40, Lemma 2.6]. For viscosity supersolution $-\mathbf{1}_E$ we can take $K = Lip(f)$ in (7.2).

We cannot in general expect uniqueness of a solution in the sense of Definition 7.1 since there may occur fattening phenomena. The comparison principle between superflows and subflows requires a positive distance between initial data and therefore by itself does not provide uniqueness. The following theorem appeared in [40].

Theorem 7.2 (c.f. [40, Theorem 2.7])**.** *Let E be a superflow with initial datum E^0 and F be a subflow with initial datum F^0 in the sense of Definition 7.1. If* $\operatorname{dist}^{M^\circ}(E^0, (F^0)^{\mathsf{c}}) =: \delta > 0$*, then*

$$\operatorname{dist}^{M^\circ}(E(t), F(t)^{\mathsf{c}}) \geq \delta e^{-Kt} \qquad \text{for all } t \geq 0,$$

where $K > 0$ is the constant in (7.2) *for both E and F.*

To obtain uniqueness, [40] introduce the associated level-set flow.

Definition 7.3. *Let u^0 be a uniformly continuous function on $\mathbb{R}^n$. We say that a lower semicontinuous function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a* level-set supersolution *corresponding to* (7.1) *with initial datum u^0 if $u(\cdot, 0) \geq u^0$ and if for a.e. $\lambda \in \mathbb{R}$ the closed sublevel set $\{u \leq \lambda\}$ is a superflow of* (7.1) *in the sense of Definition 7.1 with initial datum $\{u_0 \leq \lambda\}$.*

Similarly, an upper semicontinuous function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a level-set subsolution *corresponding to* (7.1) *with initial datum u^0 if $-u$ is a level-set supersolution in the previous sense, with initial datum $-u_0$ and with f replaced by $-f$.*

A continuous function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a level-set solution *corresponding to* (7.1) *with initial datum u^0 if it is both a level-set supersolution and level-set subsolution with the same initial datum.*

Our terminology here is different from that in [83, Chapter 5]. A superflow here is called a set-theoretic supersolution in [83]. A level set supersolution in [83] is a superflow given by sublevel set of a continuous level-set supersolution.

The following comparison theorem was proven in [40].

Theorem 7.4 (c.f. [40, Theorem 2.5]). *Let u^0, v^0 be uniformly continuous functions on $\mathbb{R}^n$ and let u, v be respectively a level-set subsolution with initial datum u^0 and a level-set supersolution with initial datum v^0, in the sense of Definition 7.3. If $u^0 \leq v^0$ then $u \leq v$.*

The main idea using Theorem 7.2 to prove Theorem 7.4 is that due to the uniform continuity, the superflow $\{u \geq \lambda_1\}$ and the superflow $\{v \leq \lambda_2\}$ for $\lambda_1 > \lambda_2$ are initially separated by a positive distance so that Theorem 7.2 applies.

It remains to establish the existence of the level-set solutions. In the smooth case, the notion in the sense of Definition 7.3 is equivalent to the standard notion of viscosity solutions. In general, an approximation by a sequence of smooth anisotropies M_n, σ_n and a stability result established in [41, Theorem 2.8] allows to construct a level-set solution as the limit of viscosity solutions. However, the stability result requires that the approximating sequence M_n is *uniformly σ_n regular*, that is, it is required that there exists $\varepsilon_0 > 0$ such that

$$M_n = M_{0,n} + \varepsilon_0 \sigma_n$$

for all n for some convex functions $M_{0,n}$. Or equivalently, the Wulff shapes W_{M_n} is must satisfy interior W_σ condition uniformly in n. Intuitively, if M is σ regular the level sets of $d := \mathrm{dist}^{M^\circ}(\cdot, E)$ have σ-curvature bounded by C/d for some constant $C > 0$.

In particular, this stability result is only able to construct level-set solutions in the sense of Definition 7.3 if M is σ-regular. Therefore the authors of [41] propose a definition of a solution via approximation.

Definition 7.5 (c.f. [41, Definition 3.6]). *A continuous function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a* solution via approximation *to the level set flow corresponding to (7.1) with initial datum u^0 if there exists a sequence $\{M_n\}$ of σ-regular mobilities such that $M_n \to M$ and, denoting u_n the unique level-set solution of (7.1) with mobility M_n and initial datum u^0, we have $u_n \to u$ locally uniformly in $\mathbb{R}^n \times [0, \infty)$.*

Such a solution always exists and is independent of the approximating sequence $\{M_n\}$.

Theorem 7.6 (c.f. [41, Theorem 3.7]). *Let u^0 be a uniformly continuous function on $\mathbb{R}^n$. There exists a unique solution u in the sense of Definition 7.5 with initial datum u^0.*

Alternatively, the level-set flow solutions in Definition 7.3 and the solutions via approximation Definition 7.5 can be constructed using a minimizing movement scheme; see [40] and the discussion in Section 6.4. To be more precise, for given initial data u^0 one can define the level set discrete evolution $u_h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as

$$u_h(x,t) := \inf\{\lambda \in \mathbb{R} : x \in E_{\lambda,h}(t)\},$$

where $E_{\lambda,h}(t)$ is the discrete evolution given by Chambolle's scheme in (6.15) with $E_0 := \{u_0 \le \lambda\}$. The following result was proved in [40, Th. 5.7].

Theorem 7.7. *Let u^0 be a uniformly continuous function on $\mathbb{R}^n$. The unique solution of (7.1) in Theorem 7.6 is the locally uniform limit in $\mathbb{R}^n \times [0,+\infty)$ as $h \to 0^+$ of the level set minimizing movements u_h.*

Here are the types of solutions that are currently available if velocity law is linear in curvature, i.e., of the form (7.1), and the initial data u^0 is constant outside of a bounded ball:

- σ smooth, M arbitrary: classical viscosity solutions [45]
- σ purely crystalline, M arbitrary: crystalline viscosity solutions [93]
- σ arbitrary, M is σ-regular: level-set solutions [40, 41]
- M, σ arbitrary: solutions via approximation [40, 41]

If the velocity law is not linear in curvature, only the viscosity solutions are currently available. On the other hand, the latter two notions apply also to general uniformly continuous initial data.

If the law is linear in the curvature, σ is purely crystalline and u^0 is constant outside of a large ball, so that the notions of crystalline viscosity solutions and solutions via approximation both apply, they also give the same solutions. This can be seen by applying stability properties under the approximation of σ by smooth σ_n.

Notion of solutions	σ	M
classical viscosity solutions [45]	C^2	any+
crystalline viscosity solutions [93]	purely crystalline	any+
level-set solutions [40, 41]	any	σ-regular
solutions via approximation [40, 41]	any	any

any+: allows any nonnegative function, not just anisotropies.

8. Some numerics

The study of the crystalline mean curvature flow using numerical methods goes back to the seminal work of J. E. Taylor, who developed the *crystalline algorithm* based on the polygonal flow in Section 3 in both two and three dimensions [157, 159], including spiral growth in two dimensions and observation of possible facet breaking in three dimensions. Examples of facet breaking were further numerically investigated in [138].

In higher dimension, the crystalline algorithm is limited to evolutions in which topological changes or facet breaking do not occur, or the result of facet breaking can be computed and produces facets with somewhat simple topology. In a more general situation, the level set method is popular to track the evolution past singularities. However, the level set equation for the crystalline mean curvature is rather singular and so its direct use is limited.

An anisotropic version of the Allen–Cahn equation was used to approximate the crystalline mean curvature flow in three dimensions in [147]. In particular, an example of facet bending was demonstrated.

A. Chambolle reformulated the minimizing movements scheme of [2] and [128] for anisotropic mean curvature flow in terms of the signed distance function as the level set function and proposed a numerical method to solve the resulting minimization problem in [39] (see Section 6.4 for more details). In [140] it was observed that the minimization problem in Chambolle's scheme can be solved efficiently using the split-Bregman method for the total variation minimization [97], and presented computational results for two dimensional crystalline mean curvature flow. However, the method easily generalizes to any dimension; see [148] for computational results for three dimensional evolutions.

It is also possible to regularize the crystalline anisotropy and consider the almost-crystalline but smooth anisotropic mean curvature flow, with many numerical methods available. One way to approximate the smooth anisotropic mean curvature flow numerically is using the Allen–Cahn equation (6.14) with double obstacle potentials (see §6.4) [13, 14, 15]. For estimates of the Allen–Cahn approximation see for example [53]. Another possibility is to track the evolving surface explicitly using a parametric approach [51, 16, 17].

For an extensive review of the early numerical approaches see [48].

9. Volume-preserving and fourth-order problems

9.1. **Volume preserving flow.** In many applications it is important to impose that the volume of the set surrounded by the evolving surface is preserved. Examples include crystal growth, droplet motion and bubbles. A common way to achieve this for the mean curvature flow is to add a Lagrange multiplier to the velocity law. Consider a family of hypersurfaces $\{\Gamma_t\}$ with $\Gamma_t = \partial\Omega_t$ for some evolving set $\{\Omega_t\}$ that evolves with the velocity law

$$V = g(\nu, \kappa_\sigma + \lambda) \qquad \text{on } \Gamma_t = \partial\Omega_t.$$

Here the forcing term $\lambda = \lambda(t)$ is chosen so that

$$|\Omega_t| = |\Omega_0| \qquad t \geq 0.$$

If $\{\partial\Omega_t\}$ is sufficiently smooth, we have

$$\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} V \, d\mathcal{H}^{n-1},$$

and $\lambda(t)$ must be chosen so that

$$\int_{\partial\Omega_t} g(\nu, \kappa_\sigma + \lambda(t)) \, d\mathcal{H}^{n-1} = 0, \qquad t \geq 0.$$

In general, the regularity of λ is not clear.

The problem has been studied in the case of linear dependence on κ_σ,

$$V = M(\nu)(\kappa_\sigma + \lambda).$$

For convex initial data, the existence of solutions and convergence to the Wulff shape W_σ was shown in [8] for smooth σ, and in [19] for nonsmooth σ, generalizing the classical result for the isotropic mean curvature flow of [104]. For a planar crystalline flow, a similar result has been proved by [168]. Moreover, it approximates corresponding smooth problems as proved in [164].

For general initial data, the existence of solutions still remains mostly open. In the isotropic case, global existence results are available under a certain energy convergence assumption [135, 125].

One can also consider initial data for which topological changes do not occur like star-shaped sets in the isotropic case [118] or sets that satisfy a certain reflection symmetry property in the anisotropic case including some crystalline flow [119].

9.2. **Fourth-order problem.** We begin with a fourth-order model to describe a relaxation process of a crystal surface by surface diffusion under the roughening temperature, which is proposed by [152] as mentioned in Section 2. It is explicitly written as

$$w_t = -\Delta\big(\operatorname{div}(\nabla w/|\nabla w|) + \beta \operatorname{div}(|\nabla w|\nabla w)\big)$$

with $\beta > 0$, where $w(x,t)$ represents the height of a crystal at x and at time t. Fortunately, this can be handled by the theory of maximal monotone operators [74], [86]. Let $H^1_{\mathrm{av}}(\mathbb{T}^n)$ denote the space of average-free H^1 functions equipped with the inner product

$$(f,g)_1 := \sum_{i=1}^n \int_{\mathbb{T}^n} \partial_{x_i} f \partial_{x_i} g \, dx.$$

In other words,

$$H^1_{\mathrm{av}}(\mathbb{T}^n) = \left\{ f \in L^2(\mathbb{T}^n) \;\middle|\; \|f\|_{H^1} = (f,f)_{H^1}^{1/2} < \infty,\ \int_{\mathbb{T}^n} f dx = 0 \right\}.$$

It is of course a Hilbert space. This space is densely embedded in

$$L^2_{\mathrm{av}}(\mathbb{T}^n) = \left\{ f \in L^2(\mathbb{T}^n) \;\middle|\; \int_{\mathbb{T}^n} f dx = 0 \right\}.$$

The dual space of H^1_{av} (under L^2 pairing) is denoted by H^{-1}_{av}. The canonical isomorphism from H^1_{av} to H^{-1}_{av} is denoted by $-\Delta$ and it agrees with the usual minus Laplacian for distributions. The space $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ is a Hilbert space equipped with the inner product

$$(f,g)_{-1} := \left\langle (-\Delta)^{-1} f, g \right\rangle,$$

where $\langle\ ,\ \rangle$ denotes a canonical pairing of H^1_{av} and H^{-1}_{av}. This $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ is our basic Hilbert space. We set energy

$$\mathcal{E}_{\beta,p}(w) := \int_{\mathbb{T}^n} |\nabla w| + \frac{\beta}{p} \int_{\mathbb{T}^n} |\nabla w|^p dx$$

with $p > 1$, $\beta \geq 0$. We consider the gradient flow of $\mathcal{E}_{\beta,p}$ in $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$, i.e.,

$$w_t \in -\partial \mathcal{E}_{\beta,p}(w). \tag{9.1}$$

Formally, this is an equation

$$w_t = -\Delta\Big(\operatorname{div}(\nabla w/|\nabla w|) + \beta \operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big)\Big).$$

If $\beta = 0$, this is nothing but the fourth-order total variation flow. A general theory guarantees the global-in-time existence of a solution to (9.1) with $\beta \geq 0$, $p > 1$ for any initial data $w_0 \in H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ since $\mathcal{E}_{\beta,p}$ is a lower semicontinuous convex functional on $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$. The important difference between second-order and fourth-order is that in the latter the comparison

principle fails. Here is an example for the case $\beta = 0$, which implies that the comparison principle should not hold.

Theorem 9.1 ([74]). *For the fourth-order total variation flow (9.1) ($\beta = 0$), the solution may become discontinuous in space even if the initial data is Lipschitz continuous.*

In [74], this is proved by giving an explicit example for $n = 1$, which works for general n. For the second-order problem, the comparison principle yields Lipschitz preserving property. Indeed, if the initial data w_0 is L-Lipschitz, then

$$w_0(x) \le w_0(x+h) + Lh =: w_{0h}.$$

The solution starting with w_{0h} is $w(x+h,t)+Lh$. If the comparison principle were valid, we would have

$$w(x,t) \le w(x+h,t) + Lh.$$

Similarly,

$$w(x,t) \ge w(x+h,t) - Lh,$$

so we would have $|w(x,t) - w(x+h,t)| \le Lh$. Theorem 9.1 shows that the comparison principle fails for (9.1) with $\beta = 0$.

Note that for $\beta > 0$, $w(\cdot,t)$ is spatially continuous for $n = 1$ since $\mathcal{E}_{\beta,p}(w) < \infty$ implies continuity.

There is a characterization of the subdifferential $\partial\mathcal{E}_{\beta,p}$ in $H^{-1}_{\mathrm{av}}(\mathbb{T}^n)$ or similar space see [116], [117] for $\beta > 0$ and [86] for $\beta = 0$. The minimal section is also calculated in [116] and [74] in the case $n = 1$; for radial case with $\beta > 0$, see [117]. There are a few differences between second-order and fourth-order problem. First, the value of $\partial^\circ\mathcal{E}_{\beta,p}$ on a facet is not determined in a neighborhood of a facet in fourth-order problem. This is in some sense expected because of a "nonlocal property" of a norm on H^{-1}_{av}. Second, the value of $\partial^\circ\mathcal{E}_{\beta,p}$ may contain δ-type function ($n = 1$), which yields instant discontinuity of a solution in Theorem 9.1.

Of course, there are several common properties between second-order and fourth-order problems. For example, the solution will stop to move in finite time. In fourth-order problems, it is only known for $n = 1, 2, 3, 4$. Let $T_*(w_0)$ be the extinction time of the solution of (9.1), i.e.,

$$T_*(w_0) = \sup\{t \in \mathbb{R} \mid w(x,t) \not\equiv 0\}.$$

Theorem 9.2 ([86]). *Let w be the solution of (9.1) with initial data $w_0 \in H^{-1}_{\mathrm{av}}$. There exists a constant C depending only on ω_i and n ($\mathbb{T}^n = \Pi_{i=1}^n(\mathbb{R}/\omega_i\mathbb{Z})$)*

(independent of dilation) such that

$$T_*(w_0) \le C\|w_0\|_{H^{-1}_{\mathrm{av}}} \quad \textit{for} \quad n = 4$$

$$T_*(w_0) \le \frac{\|w_0\|_X}{a}\left(\left(1 + \frac{a\|w_0\|^\alpha_{H^{-1}_{\mathrm{av}}}}{C\|w_0\|^\alpha_X}\right)^{1/\alpha} - 1\right) \quad \textit{for} \quad 1 \le n \le 4,\ 1 \le p \le \infty$$

with $\theta \in \left(\frac{1}{2}, 1\right]$ *satisfying* $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)(3 + n/p)$, *where* $a = (\omega_1 \cdots \omega_N)^{1/p}$, $\alpha = 2 - 1/p$ *and* $\|w_0\|_X = \|(-\Delta)^{-1}w_0\|_{\dot{W}^{-1,p}}$.

Here, $\dot{W}^{-1,p}$ is the dual of the homogeneous Sobolev space $\dot{W}^{1,p}$, i.e.,

$$\|f\|_{\dot{W}^{-1,p}} = \sup\left\{\int_{\mathbb{T}^n} f\varphi\, dx \;\middle|\; \varphi \in C^\infty(\mathbb{T}^n),\ \|\nabla\varphi\|_{L^{p'}} \le 1\right\}, \quad 1/p + 1/p' = 1.$$

The proof for $n = 4$ is easy, so we give it here for $\beta = 0$; the case $\beta > 0$ can be proved essentially in the same way. We multiply the equation

$$w_t = (-\Delta)\,\mathrm{div}\,(\nabla w/|\nabla w|)$$

with $(-\Delta)^{-1}w$ and integrate in space to get a dissipation identity

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{-1}_{\mathrm{av}}} = \int_{\mathbb{T}^n} |\nabla w| \tag{9.2}$$

since $(u, v)_{-1} = \langle(-\Delta)^{-1}u, v\rangle$. In the case $n = 4$ and $\theta = 1$, by the Sobolev and the Calderón-Zygmund inequality for $\nabla(-\Delta)^{-1/2}$, we have

$$\|w\|_{H^{-1}_{\mathrm{av}}} = \left\|(-\Delta)^{-1/2}w\right\|_{L^2} \le A'\left\|(-\Delta)^{-1/2}w\right\|_{L^p} \le A_p\|w\|_{L^p}, \quad 1/2 = 1/p - 1/4$$

for some constants A' and A_p. Again by the Sobolev inequality, there is a constant S satisfying

$$\|w\|_{L^{4/3}} \le S\int_{\mathbb{T}^n} |\nabla w|.$$

We now conclude that

$$\|w\|_{H^{-1}_{\mathrm{av}}} \le A_{4/3}S\int_{\mathbb{T}^n} |\nabla w|.$$

Thus we conclude

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{-1}_{\mathrm{av}}} \le -(A_{4/3}S)^{-1}\|w\|_{H^{-1}_{\mathrm{av}}},$$

which yields $T_*(w_0) \le C\|w_0\|_{H^{-1}_{\mathrm{av}}}$ with $C = A_{4/3}S$. For general case, we establish an interpolation inequality

$$\|w\|_{H^{-1}_{\mathrm{av}}} \le C\left\|(-\Delta)^{-1}w\right\|^{1-\theta}_{\dot{W}^{-1,p}}\left(\int_{\mathbb{T}^n} |\nabla w|\right)^\theta$$

and a rough growth estimate for a weaker norm

$$\frac{d}{dt}\left\|(-\Delta)^{-1}w\right\|_{\dot{W}^{-1,p}} \le a^{1/p}.$$

We then apply these inequalities to the dissipation identity (9.2) to get the desired estimate. For details, see [86], [87]. Combining a dissipation identity, an interpolation inequality and a growth of a weaker norm is also a key idea to estimate the coarsening rate in a surface diffusion flow as studied in [121].

There are several numerical studies for the above fourth-order singular diffusion equations. A numerical computation for $\beta > 0$, $p = 3$ is done by [122]. Their numerical scheme regularizes the singularity. A duality based numerical scheme which applies the forward-backward splitting has been proposed in [89]. A Bregman method is adjusted to the fourth-order problem by [94], where the singularity at $\nabla w = 0$ is not regularized.

We are interested in a polygonal flow by surface diffusion. Formally, a typical example is $V = -\Delta\kappa_\sigma$ when σ is crystalline. In [37] evolution by polygonal flow is proposed and there are several numerical tests. However, there is no general notion for a solution of closed curves. It is not clear what class of polygonal flows is preserved during evolution. Recently, in [76] it is shown that there is a special class of periodic piecewise linear graph-like curves which is preserved under the evolution provided that the problem is written as a gradient flow of a lower semicontinuous convex function.

If the dependence on κ_σ is nonlinear like in (2.3), no notion of a general solution is known. By studying a special solution of (2.3), a new phenomenon is found in [127] with discussion on a relation with a step motion. There is a numerical work to calculate (2.3) in [46].

Recently, the fourth-order total variation flow equation is studied in $\mathbb{R}^n$ [88]. In this case, the definition of a solution itself is not direct, especially for $n = 1, 2$. A notion of calibrability is introduced there, and it is shown that all balls are calibrable. However, unlike in the second-order total variation flow, the outside of a ball is calibrable if and only if $n \neq 2$. If $n \neq 2$, all annuli are calibrable. However, in the case $n = 2$, an annulus is calibrable if and only if it is sufficiently thin. Moreover, for $n = 1, 2$, the solution stays non-zero for all time if the initial data is a characteristic function of a ball; the finite extinction property does not hold in the case of $\mathbb{R}^n$ $(n = 1, 2)$ [88].

Appendix A. A direct proof of properties of Ψ

In this section we give a direct proof of the convexity and lower semicontinuity of Ψ used in Theorem 5.1, defined in (5.5) as

$$\Psi(v) := \inf\{\|\sigma^\circ(z)\|_\infty \mid v = -\operatorname{div} z,\ z \in L^\infty(\mathbb{T}^n)\}, \qquad v \in L^2(\mathbb{T}^n).$$

Lemma A.1. *Ψ is convex and lower semicontinuous. Moreover, if $\Psi(v) < \infty$ for some $v \in L^2(\mathbb{T}^n)$ then the infimum is attained and therefore it is a minimum.*

Proof. Step 1 We first show the lower semicontinuity of Ψ. Suppose that $v_k \to v$ in $L^2(\mathbb{T}^n)$ and $L := \liminf_{k\to\infty} \Psi(v_k) < \infty$. By selecting a subsequence (not relabeled), we can assume $\Psi(v_k) < \infty$ for all k and $\lim_{k\to\infty} \Psi(v_k) = L$. By definition of Ψ we can fix z_k with $v_k = -\operatorname{div} z_k$ and $\|\sigma^\circ(z_k)\| < \Psi(v_k) + \frac{1}{k}$ for every k. Since $\|z_k\|_\infty \le C \|\sigma^\circ(z_k)\|_\infty$ for some $C > 0$, $(z_k)_{k\in\mathbb{N}}$ is bounded in L^∞ and hence we can take a further subsequence (not relabeled) such that $z_k \to z$ weakly* in L^∞ to some vector field z. Since $v_k = -\operatorname{div} z_k$, we have

$$\int z_k \cdot \nabla\phi \, dx = \int v_k \phi \, dx \qquad \text{for all } k \in \mathbb{N},\ \psi \in C_c^\infty(\mathbb{T}^n).$$

Sending $k \to \infty$ we see that $v = -\operatorname{div} z$ in the sense of distributions. By Lemma A.2 below, we have

$$\Psi(v) \le \|\sigma^\circ(z)\| \le \liminf_{k\to\infty} \|\sigma^\circ(z_k)\| = \liminf_{k\to\infty} \Psi(v_k).$$

Step 2 The fact that the infimum is attained when $\Psi(v) < \infty$ can be proved by a direct method of the calculus of variations. If $(z_k)_{k\in\mathbb{N}}$ is a minimizing sequence, as discussed in Step 1, by compactness we may assume that it has an L^∞ weak* limit z by taking a subsequence. Step 1 also shows the lower semicontinuity in this weak* topology. Thus, z must be a minimizer.

Step 3 Convexity of Ψ is straightforward. Take v_1, v_2 with $\Psi(v_i) < \infty$. By Step 2, we can choose minimizing z_i with $v_i = -\operatorname{div} z_i$ and $\Psi(v_i) = \|\sigma^\circ(z_i)\|_\infty$ for $i = 1, 2$. We have

$$-\operatorname{div}(tz_1 + (1-t)z_2) = tv_1 + (1-t)v_2,$$

and by convexity of $\|\sigma^\circ(z)\|$ in z, we have

$$\begin{aligned}\Psi(tv_1 + (1-t)v_2) &\le \|\sigma^\circ(tz_1 + (1-t)z_2)\|_\infty \\ &\le t\|\sigma^\circ(z_1)\|_\infty + (1-t)\|\sigma^\circ(z_2)\|_\infty \\ &= t\Psi(v_1) + (1-t)\Psi(v_2).\end{aligned}$$

This finishes the proof. □

In the proof above we used the following weak* lower semicontinuity of $\|\sigma^\circ(z)\|_\infty$.

Lemma A.2. *If $z_k \to z$ weakly* in $L^\infty(\mathbb{T}^n;\mathbb{R}^n)$ then*

$$\|\sigma^\circ(z)\|_\infty \le \liminf_{n\to\infty} \|\sigma^\circ(z_k)\|_\infty .$$

Proof. Step 1: We first show the following claim: If $z \in L^\infty(\mathbb{T}^n;\mathbb{R}^n)$ then

$$\|\sigma^\circ(z)\|_\infty = \sup\left\{\int z\cdot p\,dx : p\in L^1(\mathbb{T}^n;\mathbb{R}^n),\ \|\sigma(p)\|_1 \le 1\right\}.$$

To show this, we start with the observation that $\sigma^\circ(z) = \sup\{z\cdot p : \sigma(p)\le 1\} = \sup_m z\cdot p_m$, where $\{p_m\}_{m\in\mathbb{N}}$ is a dense subset of $\{p:\sigma(p)\le 1\}$.

Clearly $\|\sigma^\circ(z)\| < \infty$. Fix now $\varepsilon > 0$ and define the measurable set

$$A := \{x : \sigma^\circ(z(x)) > \|\sigma^\circ(z)\|_\infty - \varepsilon\} = \bigcup_m A_m,$$

with

$$A_m = \{x : z(x)\cdot p_m > \|\sigma^\circ(z)\|_\infty - \varepsilon\}.$$

By definition of the L^∞ norm we have $|A| > 0$ and so there exists m such that $|A_m| > 0$. Define

$$p(x) := \frac{1}{|A_m|} p_m \mathbf{1}_{A_m}(x).$$

Clearly $\int \sigma(p(x))\,dx = \sigma(p_m) \le 1$ and

$$\int z\cdot p\,dx = \frac{1}{|A_m|}\int_{A_m} z\cdot p_m\,dx \ge \|\sigma^\circ(z)\|_\infty - \varepsilon$$

by definition of A_m. Since $\varepsilon > 0$ was arbitrary, the claim is proved.

Step 2: For any $p\in L^1(\mathbb{T}^n;\mathbb{R}^n)$ we have

$$\int z_k\cdot p\,dx \le \int \sigma^\circ(z_k)\sigma(p)\,dx \le \|\sigma^\circ(z_k)\|_\infty \|\sigma(p)\|_1$$

As $z_k \to z$ weakly* in L^∞, we deduce

$$\int z\cdot p\,dx \le \|\sigma(p)\|_1 \liminf_{n\to\infty} \|\sigma^\circ(z_k)\|_\infty$$

By claim in Step 1, we have

$$\|\sigma^\circ(z)\|_\infty = \sup_{\|\sigma(p)\|_1\le 1}\int z\cdot p\,dx \le \liminf \|\sigma^\circ(z_k)\|_\infty .$$

This finishes the proof. □

Remark A.3. Note that $\mathbb{T}^n$ can be replaced by any open set $U\subset\mathbb{R}^n$ in the above lemmas. If $|U| = \infty$, one might have to replace A_m in the definition of $p(x)$ in Step 1 of proof of Lemma A.2 by its subset of a finite positive measure.

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