FINITE ELEMENT METHOD FOR LAPLACE'S/POISSON'S EQUATION IN TWO DIMENSIONS

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1. Introduction

Consider the following equation on a two-dimensional domain Ω:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

By multiplying the problem by a test function $\varphi \in C_0^\infty(\Omega)$, integrating, and using the Green's theorem, we find that u satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

$a(u, \varphi) = \int_\Omega \nabla u \cdot \nabla \varphi \, dx$ is a bounded bilinear form on $H^1(\Omega)$, while $F(\varphi) = \int_\Omega f \varphi \, dx$ is a linear form on $H^1(\Omega)$. Moreover, a is elliptic on $H^1_0(\Omega)$. Therefore by the Lax-Milgram theorem there exists a unique solution $u \in \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega\}$ that satisfies

$$a(u, \varphi) = F(\varphi) \qquad \text{for all } \varphi \in H^1_0(\Omega). \tag{1.1}$$

We use the finite element method to discretize (1.1). We use a piece-wise linear approximation on a triangulation of Ω. We first approximate the domain Ω by Ω^h that is a union of disjoint triangles T_i^h, $i = 1, \ldots, I$. We will look for an approximate solution on a finite dimensional space

$$V^h = \{v \in C(\Omega^h) : v \text{ is linear on each } T_i^h,\ v = g^h \text{ on } \partial\Omega^h\},$$

where g^h is a piece-wise linear function on $\partial\Omega^h$ that in some way approximates g. We use test functions from the space

$$V_0^h = \{v \in C(\Omega^h) : v \text{ is linear on each } T_i^h,\ v = 0 \text{ on } \partial\Omega^h\}.$$

We shall find a unique solution $u^h \in V^h$ of the problem

$$\int_{\Omega^h} \nabla u^h \cdot \nabla \varphi \, dx = \int_{\Omega^h} f \varphi \, dx \qquad \text{for all } \varphi \in V_0^h. \tag{1.2}$$

From now on we fix $h > 0$ and drop it from the notation. We convert this into a system of linear equations of the form $Ax = b$. Let $\{r_j\}_{j=1}^J$ be the nodes of the triangulation $\{T_i\}_{i=1}^I$. We suppose that vertices $j = N+1, \ldots, J$ are the boundary nodes, $r_j \in \partial\Omega$, while the vertices $j = 1, \ldots, N$ are in the interior of Ω. Every

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triangle T_i has exactly three distinct vertices $r_{j^i_k}$, $k = 1, 2, 3$. By λ^i_k we denote the barycentric coordinate on T_i with $\lambda^i_k(r_{j^i_k}) = 1$. Let us define the maps

$$\ell : \mathbb{R}^N \to V,$$
$$\ell_0 : \mathbb{R}^N \to V_0.$$

The map ℓ assigns to each vector $v \in \mathbb{R}^N$ the piece-wise linear continuous functions $\ell(v)$, given on a triangle T_i as

$$\ell(v)(x) = \sum_{k=1}^{3} \lambda^i_k v_{j^i_k},$$

where we set

$$v_j = g^h(r_j) \qquad \text{for the boundary points } j = N+1, \ldots, J.$$

Similarly, ℓ_0 assigns to each vector $w \in \mathbb{R}^N$ a piece-wise linear continuous functions $\ell_0(w)$, given on a triangle T_i as

$$\ell(v)(x) = \sum_{k=1}^{3} \lambda^i_k w_{j^i_k},$$

where we set

$$w_j = 0 \qquad \text{for the boundary points } j = N+1, \ldots, J.$$

We note that ℓ and ℓ_0 are isomorphisms. Therefore solving (1.2) is equivalent to finding $v \in \mathbb{R}^N$ that satisfies

$$\int_{\Omega^h} \nabla \ell(v) \cdot \nabla \ell_0(w)\, dx = \int_{\Omega^h} f \ell_0(w)\, dx, \qquad \text{for all } w \in \mathbb{R}^N. \tag{1.3}$$

By linearity, we can rewrite this as

$$\sum_{i=1}^{I} \int_{T^h_i} \nabla \ell(v) \cdot \nabla \ell_0(w)\, dx = \sum_{i=1}^{I} \int_{T^h_i} f \ell_0(w)\, dx, \qquad \text{for all } w \in \mathbb{R}^N.$$

We now only need to evaluate each integral on a triangle T^h_i.

2. Integration on triangles

We now evaluate the integrals on the element $T = T^i$. We drop the index i for simplicity in the following calculations. Suppose that we have a triangle T with area S, vertices r_k, barycentric coordinates λ_k, $k = 1, 2, 3$. Suppose that at the vertices the function $\ell(v)$ has values v_k, and the function $\ell(w)$ has values w_k. We evaluate

$$\int_T f \ell_0(w)\, dx = \sum_{k=1}^{3} v_k \left(\int_T f(x) \lambda_k(x)\, dx \right).$$

On the other hand, we have

$$\int_T \nabla\ell(v)\cdot\nabla\ell_0(w)\,dx = \int_T \left(\sum_{k=1}^{3} v_k\nabla\lambda_k(x)\right)\cdot\left(\sum_{k=1}^{3} w_k\nabla\lambda_k(x)\right)\,dx$$
$$= \sum_{k,l=1}^{3} v_k w_l\left(\int_T \nabla\lambda_k\cdot\nabla\lambda_l dx\right) =$$
$$= v_1 w_1\left(\int_T |\nabla\lambda_1|^2 dx\right) + 2v_1 w_2\left(\int_T \nabla\lambda_1\cdot\nabla\lambda_2\,dx\right) + \cdots$$

Therefore we get on each triangle T_i

$$\int_{T_i} \nabla\ell(v)\cdot\nabla\ell_0(w)\,dx = \sum_{k,l=1}^{3} v_{j_k^i} w_{j_l^i}\left(\int_{T_i} \nabla\lambda_k^i\cdot\nabla\lambda_l^i\,dx\right),$$
$$\int_{T_i} f\ell_0(w)\,dx = \sum_{k=1}^{3} w_{j_k^i}\left(\int_{T_i} f\lambda_k^i\,dx\right).$$

Therefore (1.3) can be evaluated as

$$\sum_{i=1}^{I}\sum_{k,l=1}^{3} v_{j_k^i} w_{j_l^i}\left(\int_{T_i} \nabla\lambda_k^i\cdot\nabla\lambda_l^i\,dx\right) = \sum_{i=1}^{I}\sum_{k=1}^{3} w_{j_k^i}\left(\int_{T_i} f\lambda_k^i\,dx\right), \tag{2.1}$$

which is of the form

$$Av\cdot w = b\cdot w \qquad \text{for all } w\in\mathbb{R}^N, \tag{2.2}$$

where the entries of the matrix $A = (a_{mn})$ and the vector $b = (b_n)$ are given by the sums of the integrals of the barycentric coordinates over triangular elements.

Since (2.2) must be satisfied for every $w\in\mathbb{R}^N$, it is equivalent to

$$Av = b.$$

The matrix is clearly symmetric, and since the form $a(\cdot,\cdot)$ is elliptic, it is also positive definite. Therefore we can apply any of the available solvers for linear systems of this form.

3. Integration of barycentric coordinates

We now evaluate the integrals of barycentric coordinates on a triangle T in (2.1). Suppose that we have a triangle T with area S, vertices r_k, barycentric coordinates λ_k, and length of the side opposing vertex r_k denoted by d_k, $k = 1, 2, 3$. We therefore need to understand how to evaluate the integrals $\int_T \nabla\lambda_k\cdot\nabla\lambda_l\,dx$.

We have the well-known formula

$$\int_T \lambda_1^p\lambda_2^q\lambda_3^r\,dx = \frac{2p!q!r!}{(p+q+r+2)!},$$

for $p, q, r \in \mathbb{N}\cup\{0\}$.

Now we consider the gradients of λ_k. Since λ_k are linear functions, $\nabla\lambda_k$ are constant vectors. There are many ways how they can be computed. But since the barycentric coordinates have simple geometric interpretation, it makes sense to derive the integrals from geometry.

Let h_k be the height of the triangle from vertex r_k. We have

$$h_k = \frac{2S}{d_k}.$$

Note that λ_1 is zero on the edge (r_2, r_3), etc. Therefore $\nabla\lambda_1$ is perpendicular to (r_2, r_3), pointing inside the triangle. Moreover, λ_k changes from zero on the side opposing r_k to one at r_k. Since the orthogonal distance between r_k and its opposing side is h_k, the length of $|\nabla\lambda_k|$ is given by

$$|\nabla\lambda_k| = \frac{1}{h_k} = \frac{d_k}{2S}.$$

Let us now consider the inner product $\nabla\lambda_1 \cdot \nabla\lambda_2$. Since both vectors are orthogonal to the associated edges, pointing inward the triangle, the angle between them is equal to the exterior angle α_3' at the vertex r_3. By the law of cosines, we have

$$\cos\alpha_3' = \frac{d_3^2 - d_1^2 - d_2^2}{2d_1d_2}.$$

Therefore we have

$$\nabla\lambda_1 \cdot \nabla\lambda_2 = |\nabla\lambda_1||\lambda_2|\cos\alpha_3' = \frac{d_1}{2S}\frac{d_2}{2S}\frac{d_3^2 - d_1^2 - d_2^2}{2d_1d_2} = \frac{d_3^2 - d_1^2 - d_2^2}{8S^2}.$$

We therefore have

$$\int_T |\nabla\lambda_k|^2 \, dx = \frac{d_k^2}{4S}, \tag{3.1}$$

$$\int_T \nabla\lambda_1 \cdot \nabla\lambda_2 \, dx = \frac{d_3^2 - d_1^2 - d_2^2}{8S}. \tag{3.2}$$

4. The linear system

We now finally find the entries of A and b in the system (2.2). Note that

$$Av \cdot w = \sum_{m,n=1}^{N} a_{mn} v_n w_m, \qquad b \cdot w = \sum_{m=1}^{N} b_m w_m.$$

By analyzing the formula (2.1), we can find the formulas for a_{mn} and b_m using (3.1). The entry a_{mn} contains the coefficients in from of the terms containing $v_n w_m$ in the terms on the left-hand side of (2.1). Similarly, b_m contains the coefficients in the terms from the right-hand side of (2.1) and some boundary terms of the left-hand side, containing only w_n, but no v_m for any $m = 1, \dots, N$.

Let us now introduce the following notation. Let $\mathcal{T}(m) \subset \mathcal{T}$ be the set of triangles containing the vertex r_m. For m, n let $d(m, n) = |r_m - r_n|$ be the distance between nodes m and n. For m and $T \in \mathcal{T}(m)$ we denote $d(m, T)$ the length of the edge of T opposite to vertex r_m. $S(T)$ is the area of T.

We have

$$a_{mm} = \sum_{T \in \mathcal{T}(m)} \frac{d^2(m, T)}{4S(T)} \qquad \text{for } m = 1, \dots, N,$$

and

$$a_{mn} = \sum_{T \in \mathcal{T}(m) \cap T(n)} \frac{d^2(m, n) - d^2(m, T) - d^2(n, T)}{8S(T)}. \qquad \text{for } m, n = 1, \dots, N,\ m \neq n.$$

Terms containing the boundary values w_n with $n > N$ vanish since $w_n = 0$. On the other hand, we have to keep track of the terms containing $v_m w_n$ for $N+1 \leq m \leq J$, $1 \leq n \leq N$ and move them to the right-hand side into b_n. We have

$$b_n = \sum_{T \in \mathcal{T}(n)} \int_T f(x) \lambda_{T,r_n}(x)\, dx - \sum_{T \in \mathcal{T}(m)} \sum_{\substack{N+1 \leq m \leq J \\ r_m \in T}} g^h(r_m) \frac{d^2(m,n) - d^2(m,T) - d^2(n,T)}{8S(T)}.$$