FINITE ELEMENT METHOD FOR LAPLACE’S/POISSON’S EQUATION IN TWO DIMENSIONS

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1. Introduction

Consider the following equation on a two-dimensional domain Ω:
\[ \begin{align*}
-\Delta u &= f \quad \text{in } \Omega, \\
 u &= g \quad \text{on } \partial \Omega.
\end{align*} \]

By multiplying the problem by a test function \( \varphi \in C_0^\infty(\Omega) \), integrating, and using the Green’s theorem, we find that \( u \) satisfies
\[ \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega). \]

\[ a(u, \varphi) = \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx \] is a bounded bilinear form on \( H^1_0(\Omega) \), while \( F(\varphi) = \int_{\Omega} f \varphi \, dx \) is a linear form on \( H^1(\Omega) \). Moreover, \( a \) is elliptic on \( H^1_0(\Omega) \). Therefore by the Lax-Milgram theorem there exists a unique solution \( u \in \{ v \in H^1(\Omega) : v = g \text{ on } \partial \Omega \} \) that satisfies
\[ a(u, \varphi) = F(\varphi) \quad \text{for all } \varphi \in H^1_0(\Omega). \]

We use the finite element method to discretize (1.1). We use a piece-wise linear approximation on a triangulation of \( \Omega \). We first approximate the domain \( \Omega \) by \( \Omega^h \) that is a union of disjoint triangles \( T^h_i, i = 1, \ldots, I \), see Figure 1. We will look for

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\[ \begin{align*}
\text{(a)} & \quad \Omega \subseteq \Omega^h, \\
\text{(b)} & \quad \Omega \not\subseteq \Omega^h.
\end{align*} \]

Figure 1. Approximation of the domain \( \Omega \) by domain \( \Omega^h \). (a) If \( \Omega \) can has a polygonal boundary, we can just take \( \Omega^h \). (b) If \( \Omega \) has a curved boundary so that it cannot be split into triangles, we have to approximate it by a \( \Omega^h \neq \Omega \) with a polygonal boundary.
an approximate solution on a finite dimensional space

\[ V^h = \{ v \in C(\Omega^h) : v \text{ is linear on each } T^h_i, \; v = g^h \text{ on } \partial \Omega^h \}, \]

where \( g^h \) is a piece-wise linear function on \( \partial \Omega^h \) that in some way approximates \( g \).

We use test functions from the space

\[ V^h_0 = \{ v \in C(\Omega^h) : v \text{ is linear on each } T^h_i, \; v = 0 \text{ on } \partial \Omega^h \}. \]

We shall find a unique solution \( u^h \in V^h \) of the problem

\[ \int_{\Omega^h} \nabla u^h \cdot \nabla \varphi \, dx = \int_{\Omega^h} f \varphi \, dx \quad \text{for all } \varphi \in V^h_0. \quad (1.2) \]

From now on we fix \( h > 0 \) and drop it from the notation. We convert this into a system of linear equations of the form \( Ax = b \). Let \( \{r_j\}_{j=1}^J \) be the nodes of the triangulation \( \{T_i\}_{i=1}^I \). We suppose that vertices \( j = N + 1, \ldots, J \) are the boundary nodes, \( r_j \in \partial \Omega \), while the vertices \( j = 1, \ldots, N \) are in the interior of \( \Omega \). Every triangle \( T_i \) has exactly three distinct vertices \( r^i_{jk}, k = 1, 2, 3 \). By \( \lambda^i_k \) we denote the barycentric coordinate on \( T_i \) with \( \lambda^i_k(r^i_{jk}) = 1 \). Let us define the maps

\[ \ell : \mathbb{R}^N \to V, \]

\[ \ell_0 : \mathbb{R}^N \to V_0. \]

The map \( \ell \) assigns to each vector \( v \in \mathbb{R}^N \) the piece-wise linear continuous functions \( \ell(v) \), given on a triangle \( T_i \) as

\[ \ell(v)(x) = \sum_{k=1}^3 \lambda^i_k v^i_{jk}, \]

where we set

\[ v^i_j = g^h(r^i_j) \quad \text{for the boundary points } j = N + 1, \ldots, J. \]

Similarly, \( \ell_0 \) assigns to each vector \( w \in \mathbb{R}^N \) a piece-wise linear continuous functions \( \ell_0(w) \), given on a triangle \( T_i \) as

\[ \ell_0(w)(x) = \sum_{k=1}^3 \lambda^i_k w^i_{jk}, \]

where we set

\[ w^i_j = 0 \quad \text{for the boundary points } j = N + 1, \ldots, J. \]

We note that \( \ell \) and \( \ell_0 \) are isomorphisms. Therefore solving (1.2) is equivalent to finding \( v \in \mathbb{R}^N \) that satisfies

\[ \int_{\Omega^h} \nabla \ell(v) \cdot \nabla \ell_0(w) \, dx = \int_{\Omega^h} f \ell_0(w) \, dx, \quad \text{for all } w \in \mathbb{R}^N. \quad (1.3) \]

By linearity, we can rewrite this as

\[ \sum_{i=1}^I \int_{T^h_i} \nabla \ell(v) \cdot \nabla \ell_0(w) \, dx = \sum_{i=1}^I \int_{T^h_i} f \ell_0(w) \, dx, \quad \text{for all } w \in \mathbb{R}^N. \]

We now only need to evaluate each integral on a triangle \( T^h_i \).
2. Integration on triangles

We now evaluate the integrals on the element $T = T^i$. We drop the index $i$ for simplicity in the following calculations. Suppose that we have a triangle $T$ with area $S$, vertices $r_k$, barycentric coordinates $\lambda_k$, $k = 1, 2, 3$. Suppose that at the vertices the function $\ell(v)$ has values $v_k$, and the function $\ell(w)$ has values $w_k$. We evaluate

$$\int_T f \ell_0(w) \, dx = \sum_{k=1}^{3} v_k \left( \int_T f(x) \lambda_k(x) \, dx \right).$$

On the other hand, we have

$$\int_T \nabla \ell(v) \cdot \nabla \ell_0(w) \, dx = \int_T \left( \sum_{k=1}^{3} v_k \nabla \lambda_k(x) \right) \cdot \left( \sum_{k=1}^{3} w_k \nabla \lambda_k(x) \right) \, dx = \sum_{k,l=1}^{3} v_k w_l \left( \int_T \nabla \lambda_k \cdot \nabla \lambda_l \, dx \right) = v_1 w_1 \left( \int_T |\nabla \lambda_1|^2 \, dx \right) + 2v_1 w_2 \left( \int_T \nabla \lambda_1 \cdot \nabla \lambda_2 \, dx \right) + \cdots$$

Therefore we get on each triangle $T_i$

$$\int_{T_i} \nabla \ell(v) \cdot \nabla \ell_0(w) \, dx = \sum_{k,l=1}^{3} v_{j_k} w_{j_l} \left( \int_{T_i} \nabla \lambda_{k} \cdot \nabla \lambda_{l} \, dx \right),$$

$$\int_{T_i} f \ell_0(w) \, dx = \sum_{k=1}^{3} w_{j_k} \left( \int_{T_i} f \lambda_{k} \, dx \right).$$

Therefore (1.3) can be evaluated as

$$\sum_{i=1}^{3} \sum_{k,l=1}^{3} v_{j_k} w_{j_l} \left( \int_{T_i} \nabla \lambda_{k} \cdot \nabla \lambda_{l} \, dx \right) = \sum_{i=1}^{3} \sum_{k=1}^{3} w_{j_k} \left( \int_{T_i} f \lambda_{k} \, dx \right),$$

which is of the form

$$(2.2) \quad A v \cdot w = b \cdot w \quad \text{for all } w \in \mathbb{R}^N,$$

where the entries of the matrix $A = (a_{mn})$ and the vector $b = (b_n)$ are given by the sums of the integrals of the barycentric coordinates over triangular elements.

Since (2.2) must be satisfied for every $w \in \mathbb{R}^N$, it is equivalent to

$$A v = b.$$  

The matrix is clearly symmetric, and since the form $a(\cdot, \cdot)$ is elliptic, it is also positive definite. Therefore we can apply any of the available solvers for linear systems of this form.

3. Integration of barycentric coordinates

We now evaluate the integrals of barycentric coordinates on a triangle $T$ in (2.1). Suppose that we have a triangle $T$ with area $S$, vertices $r_k$, barycentric coordinates $\lambda_k$, and length of the side opposing vertex $r_k$ denoted by $d_k$, $k = 1, 2, 3$. We therefore need to understand how to evaluate the integrals $\int_T \nabla \lambda_k \cdot \nabla \lambda_l \, dx$. 

We have the well-known formula
\[
\int_T \lambda_1^p \lambda_2^q \lambda_3^r \, dx = \frac{2^p q! r!}{(p + q + r + 2)!} S,
\]
for \( p, q, r \in \mathbb{N} \cup \{0\} \). See Section A.

3.1. Example. We have
\[
\int_T \lambda_i \, dx = S^3, \\
\int_T \lambda_i \lambda_j \, dx = S_{12} (1 + \delta_{ij}) = \begin{cases} S & i = j, \\ \frac{S}{12} & i \neq j. \end{cases}
\]

Now we consider the gradients of \( \lambda_k \). Since \( \lambda_k \) are linear functions, \( \nabla \lambda_k \) are constant vectors. There are many ways how they can be computed. But since the barycentric coordinates have simple geometric interpretation, it makes sense to derive the integrals from geometry.

Let \( h_k \) be the height of the triangle from vertex \( r_k \). We have
\[
h_k = \frac{2S}{d_k}.
\]

Note that \( \lambda_1 \) is zero on the edge \((r_2, r_3)\), etc. Therefore \( \nabla \lambda_1 \) is perpendicular to \((r_2, r_3)\), pointing inside the triangle. Moreover, \( \lambda_k \) changes from zero on the side opposing \( r_k \) to one at \( r_k \). Since the orthogonal distance between \( r_k \) and its opposing side is \( h_k \), the length of \( |\nabla \lambda_k| \) is given by
\[
|\nabla \lambda_k| = \frac{1}{h_k} = \frac{d_k}{2S}.
\]

Let us now consider the inner product \( \nabla \lambda_1 \cdot \nabla \lambda_2 \). Since both vectors are orthogonal to the associated edges, pointing inward the triangle, the angle between them is equal to the exterior angle \( \alpha_3' \) at the vertex \( r_3 \). By the law of cosines, we have
\[
\cos \alpha_3' = \frac{d_3^2 - d_1^2 - d_2^2}{2d_1d_2}.
\]

Therefore we have
\[
\nabla \lambda_1 \cdot \nabla \lambda_2 = |\nabla \lambda_1||\nabla \lambda_2| \cos \alpha_3' = \frac{d_1}{2S} \frac{d_2}{2S} \frac{d_3^2 - d_1^2 - d_2^2}{2d_1d_2} = \frac{d_3^2 - d_1^2 - d_2^2}{8S^2}.
\]

We therefore have
\[
\int_T |\nabla \lambda_k|^2 \, dx = \frac{d_k^2}{4S},
\]
\[
\int_T \nabla \lambda_1 \cdot \nabla \lambda_2 \, dx = \frac{d_3^2 - d_1^2 - d_2^2}{8S}.
\]

3.2. Partial derivatives of \( \lambda_i \). We can use the idea of for computing the gradient of the barycentric coordinates to compute their partial derivatives \( \lambda_{i,x_1}, \lambda_{i,x_2} \). Indeed, since \( \nabla \lambda_i \) is just a vector orthogonal to the edge opposite to vertex \( i \), we can simply rotate that edge to get the direction of \( \nabla \lambda_i \). We then only need to normalize this vector to have length \( h_i = \frac{2S}{d_i} \). This gives us a formula
\[
\lambda_{i,x_1} = \pm \frac{r_{i',2} - r_{i'',2}}{h_k d_k} = \pm \frac{r_{i',2} - r_{i'',2}}{2S},
\]
where \( i', i'' \) are the two other vertices than \( i \). The sign must be chosen so that \( \nabla \lambda \) point correctly inside the triangle.
4. The linear system

We now finally find the entries of $A$ and $b$ in the system (2.2). Note that

$$Av \cdot w = \sum_{m,n=1}^{N} a_{mn} v_n w_m, \quad b \cdot w = \sum_{m=1}^{N} b_m w_m.$$  

By analyzing the formula (2.1), we can find the formulas for $a_{mn}$ and $b_m$ using (3.1).

The entry $a_{mn}$ contains the coefficients in from of the terms containing $v_n w_m$ in the terms on the left-hand side of (2.1). Similarly, $b_m$ contains the coefficients in the terms from the right-hand side of (2.1) and some boundary terms of the left-hand side, containing only $w_n$, but no $v_m$ for any $m = 1, \ldots, N$.

Let us now introduce the following notation. Let $T(m) \subset T$ be the set of triangles containing the vertex $r_m$. For $m,n$ let $d(m,n) = |r_m - r_n|$ be the distance between nodes $m$ and $n$. For $m$ and $T \in T(m)$ we denote $d(m,T)$ the length of the edge of $T$ opposite to vertex $r_m$. $S(T)$ is the area of $T$.

We have

$$a_{mm} = \sum_{T \in T(m)} \frac{d^2(m,T)}{4S(T)} \quad \text{for } m = 1, \ldots, N,$$

and

$$a_{mn} = \sum_{T \in T(m) \cap T(n)} \frac{d^2(m,n) - d^2(m,T) - d^2(n,T)}{8S(T)}, \quad \text{for } m,n = 1, \ldots, N, m \neq n.$$

Terms containing the boundary values $w_n$ with $n > N$ vanish since $w_n = 0$. On the other hand, we have to keep track of the terms containing $v_m w_n$ for $N + 1 \leq m \leq J$, $1 \leq n \leq N$ and move them to the right-hand side into $b_n$. We have

$$b_n = \sum_{T \in T(n)} \int_T f(x) \lambda_{T,r_n}(x) \, dx$$

$$- \sum_{T \in T(m) \cap T(n)} \sum_{r_m \in T} g^h(r_m) \frac{d^2(m,n) - d^2(m,T) - d^2(n,T)}{8S(T)}.$$

4.1. Example. In the case of the regular grid in 2 dimensions, where every square is split into two triangles, the above formulas yield the finite difference matrix.

5. Per triangle formulas

In this section we consider an element $T$ with area $S$ and with two linear functions $u$, $v$ on $T$ with values $u^i$, $v^i$ in the vertices. We have

$$\int_T uv \, dx = \sum_{i,j=1}^{3} \left( \int_T \lambda_i \lambda_j \right) u^i v^j.$$  

Similarly,

$$\int_T \nabla u \cdot \nabla v \, dx = \sum_{i,j=1}^{3} \left( \int_T \nabla \lambda_i \cdot \nabla \lambda_j \right) u^i v^j.$$
Consider a more advance example of vector functions \( u = (u_1, \ldots, u_K) \), \( v = (v_1, \ldots, v_K) \). We have

\[
\int_T \text{div} u \text{ div} v \, dx = \sum_{i,j=1}^{3} \sum_{k,l=1}^{K} \left( \int_T \lambda_{i,x} \lambda_{j,x} \right) u_k^i v_l^j.
\]

**Appendix A. Integration of Barycentric Polynomials**

Given a \( n \)-dimensional simplex \( S \), we claim that

\[
\int_S \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_{n+1}^{p_{n+1}} \, dx = \frac{n! \prod_{i=1}^{n+1} p_i!}{(n + \sum_{i=1}^{n+1} p_i)!} \text{Vol}(S),
\]

for any \( p_i \in \mathbb{N} \cup \{0\} \).

Suppose that \( S \) is an \( n \)-dimensional simplex of nonzero volume with vertices \( v_1, \ldots, v_{n+1}, \) i.e., \( S = \text{conv}(v_1, \ldots, v_{n+1}) \). In particular, \( v_i - v_{n+1} \) are linearly independent. There is an invertible affine transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( T v_i = e_i \) and \( T v_{n+1} = 0 \). Clearly \( |\det(DT)| = \text{Vol}(v_1 - v_{n+1}, \ldots, v_n - v_{n+1})^{-1} = (n! \text{Vol}(S))^{-1} \).

Let \( \lambda_i \) be the barycentric coordinate on \( S \) such that \( \lambda_i(v_i) = 1 \). We therefore have \( \lambda_i(T^{-1}x) = x_i \) for \( 1 \leq i \leq n \) and \( \lambda_{n+1}(T^{-1}x) = 1 - \sum_{i=1}^{n} x_i \).

By change of variables formula, we have

\[
\int_S \prod_i \lambda_i^{p_i} \, dx = n! \text{Vol}(S) \int_{S(T)} \prod_i \lambda_i^{p_i} \, dx
\]

\[
= n! \text{Vol}(S) \int_{\tilde{S}} \prod_{i=1}^{n} x_i^{p_i} (1 - \sum_{i=1}^{n} x_i)^{p_{n+1}} \, dx,
\]

where \( \tilde{S} = \text{conv}(0, e_1, \ldots, e_n) \). Since

\[
I := \int_{\tilde{S}} \cdots \, dx = \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} \cdots \int_0^{1-x_1-\cdots-x_n} \cdots \, dx_n \cdots \, dx_1,
\]

we introduce the change of coordinates

\[
t_1 = x_1,
\]

\[
(1 - t_1)t_2 = (1 - x_1)t_2 = x_2,
\]

\[
(1 - t_1)(1 - t_2)t_3 = (1 - x_1 - x_2)t_3 = x_3,
\]

\[
\vdots
\]

\[
\prod_{i=1}^{n-1} (1 - t_i)t_n = (1 - \sum_{i=1}^{n-1} x_i)t_n = x_n,
\]

and we have

\[
\prod_{i=1}^{n} (1 - t_i) = (1 - \sum_{i=1}^{n} x_i).
\]
Using Fubini’s theorem and the Beta function $B$,

$$\begin{align*}
I &= \prod_{i=1}^{n} \left( \int_{0}^{1} t^{p_{i}} (1 - t)^{\sum_{j=i+1}^{n} (p_{j} + 1) + p_{n+1}} \, dt \right) \\
&= \prod_{i=1}^{n} B \left( p_{i} + 1, \sum_{j=i+1}^{n+1} (p_{j} + 1) \right) \\
&= \prod_{i=1}^{n} \frac{p_{i}! \left( \sum_{j=i+1}^{n+1} (p_{j} + 1) - 1 \right)!}{p_{i} + 1 + \sum_{j=i+1}^{n+1} (p_{j} + 1) - 1)!} \\
&= \prod_{i=1}^{n} \frac{p_{i}! \left( \sum_{j=i+1}^{n+1} (p_{j} + 1) - 1 \right)!}{\left( \sum_{j=i}^{n+1} (p_{j} + 1) - 1 \right)!} \\
&= \frac{\prod_{i=1}^{n+1} p_{i}!}{\left( \sum_{j=1}^{n+1} (p_{j} + 1) - 1 \right)!} \\
&= \frac{\prod_{i=1}^{n+1} p_{i}!}{\left( n + \sum_{j=1}^{n+1} p_{j} \right)!}
\end{align*}$$

Plugging this into (A.2), we have (A.1).