ON THE GEOMETRY OF RATE INDEPENDENT DROPLET EVOLUTION

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Abstract. We consider a toy model of rate independent droplet motion on a surface with contact angle hysteresis based on the one-phase Bernoulli free boundary problem. Taking advantage of two notions of weak solutions, energy-based and comparison-principle-based, we study the dynamic contact angle of moving contact lines and the geometry of de-pinning. We show that these two notions of solutions coincide in a star-shaped setting, where we show (almost) optimal regularity of the contact line and the convergence of a minimizing movements scheme. In a general setting, the notions differ essentially in how they handle jumps, but both are shown to satisfy a weak motion law.

Contents

1. Introduction

In this work we consider a toy model, based on the one-phase Bernoulli free boundary problem, for the rate independent motion of droplets on a solid surface with the effects of contact angle hysteresis. We study two related notions of weak solution which handle time jump discontinuities differently: an energetic notion and a notion based on comparison principle and a free boundary obstacle problem. We study the geometry of the contact line and the dynamic contact angle of moving contact lines for these solutions. Our first main result is in a star-shaped setting. We show equivalence between the typical minimizing movements energy solutions and the obstacle solutions based on a novel comparison principle. We also show the optimal $C^{1,\frac{1}{2}-}$ regularity of the contact line. Our second main result is in a general setting. We show that both energy solutions and obstacle solutions satisfy a weak motion law essentially reducing the difference of these two evolutions to

Key words and phrases. Rate-independent evolution, free boundaries, contact angle hysteresis, free boundary regularity.

their jump laws. To our knowledge these are the first rigorous results establishing these properties for a general class of weak solutions to a rate independent model of droplet evolution.

It is well understood in the physics and engineering literature that capillary droplets on many solid surfaces experience a phenomenon known as *contact angle hysteresis* [18]. Rather than a single contact angle specified by the material properties, as would be predicted by the classical Young's law in capillarity theory [32], there is a range of stable contact angles. Consequently droplets can "stick" to the solid surface; the contact line (the curve separating the wet and dry regions) does not move under small applied forces although the free surface (the liquid-air interface) does move. The origin and appropriate modeling of contact angle hysteresis is the subject of much debate in the engineering literature [18, 32].

Similar phenomena occur in the context of two phase flow in a porous medium, called capillary pressure hysteresis in this context [16, 34, 37]. Typically the flow velocity in each phase is determined by Darcy's law and the phase interface moves with velocity proportional to a pressure differential. However, when capillarity forces at the interface, where individual grains of the medium are only partially wetted, are of the same order as the pressure differential, pinning can occur. The origin of this pinning is exactly the microscopic surface roughness of the matrix medium and the associated contact angle hysteresis. The phase interface only advances or recedes when the pressure differential exceeds a certain threshold.

The present paper is on the qualitative features of a macroscopic model for these types of interface pinning phenomena. Our model is inspired by the one proposed by DeSimone, Grunewald, and Otto [17], with theory developed by Alberti and DeSimone [1].

Instead of the capillarity energy with slowly varying volume constraint, as considered in [1, 17], we study a quasi-static and rate independent evolution associated with the Alt-Caffarelli one-phase energy functional under a slowly varying Dirichlet forcing. The mathematical reason for considering the one-phase problem instead of the capillarity problem is to simplify various technical aspects of the problem, in particular giving us access to the more developed regularity theory of the one-phase free boundary problem. Specifically, the work of Chang-Lara and Savin [12] on Bernoulli obstacle problems and its extension by Ferreri and Velichkov [22] will play a major role. We emphasize that, as it concerns capillarity, the models we consider in this work are *toy models*. Nonetheless, as is typically the purpose of toy models, we expect the mathematical philosophies developed here to be relevant to the true capillarity model.

Our contribution in this work is in connecting several energetic and geometric notions of weak solution, to study the pointwise qualitative description of those solutions: (1) *global energetic solutions* (which we just refer to as "energy solutions"), (2) *obstacle evolution solutions*, and (3) *motion law solutions*, see Section 1.1 below for definitions. Each of these notions of weak solution has different advantages, and only by connecting the concepts can we obtain a more complete understanding of the model.

Energy solutions can be defined in a quite general setting, but are difficult to work with in terms of geometric properties and their jump law can be un-physical. Among the general energy solutions, minimizing movements solutions generated by a time incremental scheme behave especially well. Obstacle solutions on the

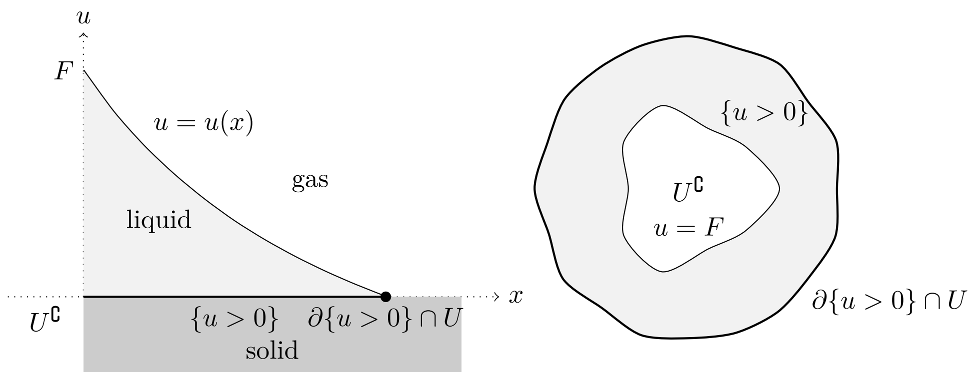

FIGURE 1. Side view (left) and the top view (right) of the setup for the one-phase free boundary problem.

other hand are easier to work with in terms of geometry, and have a desirable jump law, jumping "as late and as little as possible". However the obstacle notion only makes sense in the important special case of piecewise time monotone forcing and it is difficult to derive energetic properties. These two evolutions have different jump laws, but conceptually are linked by the idea that their motion at non-jump times is closely related. Both evolutions, at least formally, satisfy the *dynamic slope condition* which governs the continuous motion but only partially specifies the jumps.

In what follows we we introduce the notion of solutions that we will consider in the paper, and then state our main results.

1.1. **One-phase free boundary problem.** Consider a connected domain U in $\mathbb{R}^d$ with compact complement. For $u : U \to [0, \infty)$ consider the Alt–Caffarelli one-phase free boundary energy functional

$$\mathcal{J}(u) = \int_U |\nabla u|^2 + \mathbf{1}_{\{u>0\}} \, dx, \tag{1.1}$$

where $\mathbf{1}_{\{u>0\}}$ is the indicator function of the set $\Omega(u) := \{u > 0\}$.

In the context of capillarity modelling the droplet's free surface (liquid–gas) is given by the graph of u on U, and the positivity set $\Omega(u)$ is the wetted area: the contact between the liquid and solid phase; see Figure 1. The triple junction or contact line (liquid-gas-solid) is located at the free boundary $\partial\Omega(u) \cap U$. The Dirichlet energy $\int_U |\nabla u|^2 \, dx$ comes from a linearization of the surface area.

We recall that the Euler-Lagrange equation associated with the energy functional $\mathcal{J}$ is the one-phase Bernoulli free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega(u) \cap U \\ |\nabla u|^2 = 1 & \text{on } \partial\Omega(u) \cap U. \end{cases} \tag{1.2}$$

We are interested in rate independent evolutions associated with the energy functional $\mathcal{J}$, driven by a Dirichlet boundary data $F(t) : [0, T] \to (0, \infty)$ acting as the external forcing. We augment the energy functional (1.1) by adding an energy dissipation in a form of a *dissipation distance*: for any pair of sets Ω_0 and Ω_1 in $\mathbb{R}^d$

we define

$$\mathrm{Diss}(\Omega_0, \Omega_1) = \mu_+|\Omega_1 \setminus \Omega_0| + \mu_-|\Omega_0 \setminus \Omega_1|.$$

This non-symmetric distance measures the energy dissipated by the motion of the free interface under (monotone) motion of the positive phase from state Ω_0 to state Ω_1. The coefficients $\mu_+ > 0$ and $\mu_- \in (0,1)$ can be viewed as the friction forces per unit length of the free interface, respectively for advancing and receding regimes.

In what follows we will abuse notation and also write $\mathrm{Diss}(u,v) = \mathrm{Diss}(\Omega(u), \Omega(v))$ for the dissipation distance between the positive phase of two profiles u and v.

In the simple model we assume that free interface can move only if the "force" $|\nabla u|^2 - 1$ per unit length on the interface coming from the first variation of potential energy (1.1) can overcome the static friction force μ_+ or μ_-, depending whether the contact line advances or recedes. Furthermore, the scale at which the contact line moves is much faster than the scale on which we observe the state of the system, and therefore at each time t the interface is assumed to be in an equilibrium configuration and cannot move. This state $u(t)$ can be characterized as a local minimizer of the augmented energy functional

$$\mathcal{E}(u, u') := \mathcal{J}(u') + \mathrm{Diss}(u, u'), \tag{1.3}$$

that expresses the total of the potential energy $\mathcal{J}(u')$ in an alternative state u' and the energy dissipated by the friction forces on the contact line required to reach the alternative state.

The formal first variation of the augmented energy function $\mathcal{E}(u, \cdot)$ results in the pinned one-phase free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \ \{u > 0\} \cap U, \\ 1 - \mu_- \leq |\nabla u|^2 \leq 1 + \mu_+ & \text{on } \partial\{u > 0\} \cap U. \end{cases} \tag{1.4}$$

Now the gradient is allowed to take a range of values on the interface, this is the manifestation of contact angle hysteresis in this model as "gradient hysteresis".

The reason why the state might change with t is that it is subject to the forcing by Dirichlet boundary condition

$$u(t) = F(t) \qquad \text{on } \partial U$$

at each $t \geq 0$. Since we can observe the state only in an equilibrium, the evolution is rate-independent in the sense that the path does not depend on how fast F changes, that is, the time variable can be monotonically reparametrized yielding an equivalent evolution.

Varying $F(t)$ pulls up (or down) the profile $u(t)$ but the free boundary remains pinned as long as the gradient at the free boundary is within the pinning interval in (1.4). Once the gradient saturates one of the endpoints in the pinning condition (1.4) somewhere, the interface needs to move. And, indeed, the free boundary *only* advances or recedes when the gradient saturates the corresponding endpoint of the pinning interval. This heuristic suggests the *dynamic slope condition*

$$|\nabla u(t)|^2 = 1 \pm \mu_\pm \quad \text{ if } \quad \pm V_n(\Omega(u(t)), x) > 0 \quad \text{on } \ \partial\Omega(u(t)) \cap U \tag{1.5}$$

where $V_n(\Omega(u(t)), x)$ is the outward normal velocity of $\Omega(u(t))$ at $x \in \partial\Omega(u(t))$. This condition is analogous to the dynamic contact angle condition which is studied in the physics of capillarity.

1.2. **Rate independent evolution rules.** Now we proceed to detailed descriptions of the (1) energy solutions, (2) obstacle solutions, and (3) motion law solutions.

Energetic weak solutions. The view of the evolution as a continuum of stable states of the energy functional $\mathcal{E}$ motivates the first notion of solutions of the rate-independent evolution of the Bernoulli functional. However, the local minimality is strengthened to a global one, as is common in the theory of rate-independent systems [1, 30]. Additionally, an evolution between any two states at two times along the solution must be energetically allowed: the dissipated energy required to reach the second state from the first cannot be larger than the difference of potential energies of the two states together with the work done by the external forcing.

Definition 1.1. A measurable $u : [0, T] \to H^1(U)$ is a *energy solution* (E) of the quasistatic evolution problem driven by Dirichlet forcing F if the following hold:

(1) (*Forcing*) For all $t \in [0, T]$

$$u(t) = F(t) \quad \text{on } \partial U$$

(2) (*Global stability*) The solution $u(t) \in H^1(U)$ and satisfies for all $t \in [0, T]$:

$$\mathcal{J}(u(t)) \leq \mathcal{J}(u') + \mathrm{Diss}(u(t), u') \qquad \text{for all } u' \in u(t) + H^1_0(U).$$

(3) (*Energy dissipation inequality*) For every $0 \leq t_0 \leq t_1 \leq T$ it holds

$$\mathcal{J}(u(t_0)) - \mathcal{J}(u(t_1)) + \int_{t_0}^{t_1} 2\dot{F}(t)P(t)\, dt \geq \mathrm{Diss}(u(t_0), u(t_1)). \tag{1.6}$$

Here $P(t) = P(u(t)) = \int_{\partial U} \frac{\partial u(t)}{\partial n}\, dS$ is an associated pressure.

In some works, for example [1], energy solutions are required to satisfy an *energy dissipation balance* condition instead of the inequality we use. That notion is equivalent to ours, see Remark 5.8 later for more details.

The Euler-Lagrange equations for the global stability condition (2) is the pinned one-phase problem (1.4). Taking the time derivative in the energy dissipation balance, see (5.4) and (7.1) below for details of the computation, shows that smooth energy solutions satisfy the dynamic slope condition (1.5). Justifying this formal computation for general energy solutions is quite difficult and is the content of Theorem 1.6 below.

Minimizing movements scheme. The most typical way to construct an energy solution (E) is based on a time-incremental or minimizing movements scheme. Consider the time-discrete approximation scheme

$$u^k_\delta \in \operatorname{argmin} \left\{ \mathcal{J}(w) + \mathrm{Diss}(u^{k-1}_\delta, w) : w \in F(k\delta) + H^1_0(U) \right\}. \tag{1.7}$$

Using piecewise constant interpolation define, for all $t \in [0, T]$,

$$u_\delta(t) := u^k_\delta \quad \text{and} \quad F_\delta(t) = F(k\delta) \quad \text{if} \quad t \in [k\delta, (k+1)\delta). \tag{1.8}$$

The time incremental scheme does turn out to produce energy solutions via a compactness idea introduced by Mainik and Mielke [27].

Definition 1.2. Say that $u : [0, T] \to H^1(U)$ measurable is a *minimizing movements energy solution* if there is $u_\delta(t)$ solving the scheme (1.7) and a subsequence $\delta_k \to 0$ so that $u_{\delta_k}(t) \to u(t)$ in $L^2(U)$ for *every* $t \in [0, T]$.

Note that, despite the terminology, it definitely requires proof to show that minimizing movements energy solutions actually solve (E). Our results for minimizing movements solutions are stronger than for general energy solutions of (E), see Theorem 1.5.

Geometric weak solutions: obstacle solution. Next we introduce an, apparently, quite different notion of rate independent evolution associated with the one-phase problem which we call the *obstacle solution*. We will see that this notion is physically quite natural, but it is difficult to classify the evolution in terms of energetics, see Section 1.4 for discussion of possibly related energetic notions.

This notion of solution does rely on a certain monotonicity hypothesis on the forcing. More precisely, we assume that the forcing $F : [0, T] \to (0, \infty)$ is Lipschitz regular and only changes monotonicity on a finite set $Z \subset [0, T]$, that is, for any $(s, t) \subset [0, T] \setminus Z$ the forcing F is either strictly increasing or strictly decreasing on $[s, t]$.

Definition 1.3. We say that $u : [0, T] \times \overline{U} \to [0, \infty)$ is a *obstacle solution* (O) in U driven by F on ∂U if

(1) (*Initial data*) $u(0)$ is a viscosity solution of the local stability conditions (1.4).

(2) (*Dirichlet forcing*) For all $t \in [0, T]$

$$u(t) = F(t) \quad \text{on} \quad \partial U. \tag{1.9}$$

(3) (*Obstacle condition*) For every $(s, t) \cap Z = \emptyset$, so that F is monotone on $[s, t]$, $u(t)$ is the minimal supersolution of (1.4) and (1.9) above $u(s)$ when F is increasing on $[s, t]$ (resp. maximal subsolution below $u(s)$ when F is decreasing).

The notions of minimal supersolution and maximal subsolution above are in the viscosity solutions / Perron's method sense. The stability condition on the initial data is for convenience, otherwise the solution would jump at the initial time resulting in a "replacement" initial data arising from a Bernoulli obstacle problem.

Geometric weak solutions: dynamic slope condition. Although the obstacle solution (O) and the energetic solution (E) have different jump laws, our philosophy is that they should "only" differ due to the jump laws. The commonality of these two will be expressed in the notion of motion law solution. The motion law solution property imposes a local stability condition and a dynamic condition on the slope.

In this model the local evolution law can be naturally interpreted in the framework of viscosity solutions (i.e. solutions based on the comparison principle). This motivates the notion of *motion-law viscosity solution*.

Definition 1.4. $u : [0, T] \to C(\overline{U})$ is a *motion-law viscosity solution* (M) if

(1) (*Forcing*) For all $t \in [0, T]$

$$u(t) = F(t) \quad \text{on } \partial U.$$

(2) (*Local stability condition*) For all $t \in [0, T]$ the function $u(t) \in C(\overline{U})$ is a continuous viscosity solution of (1.4) in the semicontinuous envelope sense.

(3) (*Dynamic slope condition*) u is a viscosity solution of (1.5) on $[0, T] \times U$ in the semicontinuous envelope sense.

We recall viscosity solutions in Section 4, and in particular the precise meaning of the local stability condition in Definition 4.11 and the dynamic slope condition is given in Definition 4.15.

1.3. **Statements of main results.** Our first collection of results is on the uniqueness and regularity properties of the three evolutions in the case of strongly star-shaped data. Star-shapedness is a natural topological hypothesis in the context of free boundary problems which often allows for uniqueness and regularity and prevents topological changes. In our context star-shapedness also rules out jumps and so we can show a significant (but not total) overlap of the three notions of solution considered in this work.

Theorem 1.5 (see Theorem 4.17 and Theorem 6.6)**.** *Suppose that $\mathbb{R}^d \setminus U$ and Ω_0 are strongly star-shaped and bounded, Ω_0 is $C^{1,\alpha}$, and $F : [0,T] \to (0,\infty)$ is Lipschitz and only changes monotonicity finitely many times. Let u be the unique obstacle solution (O) on $[0,T]$ with a strongly-star-shaped initial data $u(0)$, and $u(0)$ satisfies (1.4). Then*

- *(i) The profile $u(t)$ is strongly star-shaped for each time, and it is the unique motion law viscosity solution (M) with initial data $u(0)$.*
- *(ii) The profile $u(t)$ and positivity set $\Omega(u(t))$ have the regularity $C^{0,1}_t C_x \cap L^\infty_t C^{1,\min\{\frac{1}{2},\alpha\}-}_x$.*
- *(iii) The unique (M) solution $u(t)$ is also the unique minimizing movement energy solution with initial data $u(0)$. Moreover the solutions of the discrete-time minimizing scheme converge uniformly to $u(t)$ with a uniform rate that only depends on F, $\mu_\pm$ and d.*

Besides the regularity theory developed in Section 3, the novel comparison principle for solutions of (M), Proposition 4.26, also plays a central role in this theorem. The convergence result for minimizing movements solutions part (iii) follows a similar idea to Chambolle's proof of convergence of the Almgren-Taylor-Wang / Luckhaus-Sturzenhecker schemes for mean curvature flow [2, 11, 26]. We establish that the discrete-time minimizing movements satisfy (M) and we use the aforementioned comparison principle.

The star-shapedness is a key hypothesis in Theorem 1.5. Perhaps most importantly, the star-shaped geometry is used to ensure that there is no jump in the obstacle solution (O). This is crucial to obtain part (iii), since the energy solutions and obstacle solutions have different jump laws, see Section 2.2. The occurrence of jumps is also exactly related to non-uniqueness of the exterior obstacle Bernoulli problem and the (M) problem. The local cone monotonicity implied by star-shapedness is also used to obtain the regularity result in (ii). Cone monotonicity allows us to show, in Section 3, that all blow-ups at the free boundary are half-plane solutions. Then we can invoke the "flat means smooth" results of [12, 22] for Bernoulli obstacle problems. All these properties are consistent with the idea that the energy landscape on the star-shaped sets has a convexity property, it would be interesting to make this precise.

Next we address the general case without the topological restrictions of the star-shaped setting. In Section 4 we show that all non-degenerate (O) solutions, regardless of geometry, satisfy (M), see Lemma 4.20. But whether general energy solutions are (M) is unclear: since jumps are now possible, (O) and (E) are typically different evolutions. Nonetheless we still show that energy solutions satisfy (M) in some sense. More precisely, our second main result says that general energy solutions still satisfy the dynamic slope condition in a geometric measure theoretic sense

Theorem 1.6 (see Proposition 5.16 and Theorem 7.1)**.** *Suppose u is an energy solution on $[0,T]$. Then*

(i) (Basic regularity properties) *The states $u(t)$ are uniformly Lipschitz and non-degenerate and $\mathcal{H}^{d-1}(\partial\Omega(t))$ is uniformly bounded in time. Also $t \mapsto \Omega(u(t))$ is in $BV([0,T];L^1(\mathbb{R}))$ and $u(t)$ has left and right limits in uniform metric at every time, denoted $u_\ell(t)$ and $u_r(t)$.*

(ii) (Upper and lower envelopes) *The upper and lower semicontinuous envelopes of u, called u^* and u_*, are themselves energy solutions and actually $u^*(t) = \max\{u_\ell(t), u_r(t)\}$ and $u_*(t) = \min\{u_\ell(t), u_r(t)\}$.*

(iii) (Dynamic slope condition a.e.) *For all $t \in [0,T]$ the function $u(t)$ satisfies the stability condition (1.4) and satisfies (in terms of u^* and u_*) the dynamic slope condition (1.5) at $\mathcal{H}^{d-1}$ almost every point of its free boundary $\partial\Omega(u(t)) \cap U$.*

The regularity properties in (i) follow from relatively standard energy arguments, we are recalling them here for context. The precise description of the semicontinuous envelopes (ii) is a somewhat unusual result in the context of literature on rate-independent motion. Our analysis of the energetic properties of the upper and lower envelopes in (ii), and see Proposition 5.16 below, is more typical of the theory of discontinuous viscosity solutions [13] and plays an important role in the proof of the dynamic slope condition.

1.4. **Discussion.** In order to contextualize our results we now explain further about the advantages and disadvantages of the different notions of solution and their interrelations.

Discussion of the energy solution notion. Energy solutions have the advantage of "baking in" the energy dissipation balance and this notion of solution generalizes better to the more physically relevant case of volume constrained evolution. There may also be advantages of the variational framework in studying questions related to homogenization.

On the other hand it is difficult to derive geometric properties of the evolution purely from the energetic structure. We are not aware of any purely energetic method to obtain higher regularity of the free boundary. Even seemingly simple properties are difficult to prove just from energetics, for example it is not clear whether arbitrary (E) solutions must respect time monotonicity of the Dirichlet forcing.

The central difficulty, in general, with energy solutions is to find a suitable globally defined dissipation distance. In the isotropic case that we consider in this paper there is a simple and natural dissipation distance. Such a quantity is not always available. For example, it is not clear how to define any global dissipation distance in the case of anisotropic media arising from periodic homogenization [9, 19, 21, 25]. Furthermore, even with a mathematically elegant choice of global dissipation distance, as we have, the energy evolution handles jump discontinuities in a "non-physical" way. This is a well-known and common shortcoming of the, so-called, global energetic solutions in the dissipative evolution framework [28].

One of main proposals, within the energetic framework, for improving the "locality" of the jumps is the notion of *balanced viscosity solution* [14, 28, 29, 31] (different from the comparison viscosity solutions we consider below) and see the book [30, Chapter 3.8.2]. This notion is based on a modification of the time incremental minimization scheme (1.7), adding an L^2-type distance term in the incremental minimization problem which strongly penalizes jumps. The balanced viscosity approach leads to a notion of energetic solution which jumps "as late and as little as possible".

In the context of the literature on rate independent motions our work is novel in that we can take advantage of certain monotonicity and comparison structures in the problem in order to (1) give a simple non-energetic characterization of the solutions which jump "as late and as little as possible", (2) display the optimal regularity of energy solutions induced by the dynamic slope condition.

Discussion of the obstacle solution notion. The obstacle solution (O) notion is well suited to study the regularity theory and qualitative geometric features of the free boundary.

In particular using this notion, in Section 3 and Section 4, we can study the shape of the free boundary as it de-pins. We will use regularity results for Bernoulli obstacle problems due to Chang-Lara and Savin [12] and very recently extended to the obstacle from below case by Ferreri and Velichkov [22]. The regularity at de-pinning turns out to be governed, at first order in the flat asymptotic expansion, by a thin obstacle problem. The thin obstacle problem is well known to have $C^{1,\frac{1}{2}}$ optimal regularity and we inherit $C^{1,\frac{1}{2}-}$ regularity for obstacle solutions in the star-shaped case, see Theorem 1.5 below.

The other advantage of the obstacle solution, even outside of the star-shaped case, is that it handles time jump discontinuities well. Specifically, the obstacle solution jumps "as late and as little as possible", which is regarded as a more physical jump condition the same as the notion of balanced viscosity solution, see for example [28] or [30, Chapter 3.8.2]. The obstacle solution dissipates the "right" amount of energy on its jumps, but does not obviously yield an energetic notion of jump dissipation. It would be interesting to study the possible connection of the balanced viscosity notion with our obstacle solution (O), but we do not address that in this work.

Finally we mention that the problem (O) has an easy uniqueness property from its definition, although establishing local conditions equivalent to the minimality / maximality properties in (O) is not so easy in general, see Section 4.

Discussion of the motion law solution notion. The concept of the motion law solution property, which we can partially but not fully justify in this work, is that the motion law viscosity solution property contains all the information in (E) or (O) *except for the jump law.* We do not have a way to make this concept fully precise, but our results are in support of this notion.

The motion-law viscosity solution property, philosophically speaking, contains significantly less information than either the notion of energy solution or the notion of obstacle solution. The (M) property only imposes a very mild condition on the jumps via (1.5). Specifically that the dynamic slope is saturated at any point that has jumped strictly inwards or outwards. This property is indeed satisfied by both energy solutions and obstacle solutions.

Despite this philosophy it is still highly nontrivial to actually prove the (M) property, especially for general energy solutions. In Lemma 4.20 we show that (O) solutions do satisfy (M). And our second main result, Theorem 1.6, shows that the dynamic slope condition of (M) at least holds in the surface measure almost everywhere sense.

1.5. **Acknowledgments.** W. Feldman was partially supported by the NSF grant DMS-2009286. I. Kim was partially supported by the NSF grant DMS-2153254. N. Pozar was partially supported by JSPS KAKENHI Kiban C Grant No. 23K03212.

1.6. **Notations and conventions.** We list several notations and conventions which will be in force through the paper.

- ▷ We call a constant *universal* if it only depends on d and $\mu_+ > 0$, $\mu_- \in (0,1)$, a constant is called *dimensional* if it only depends on the dimension.
- ▷ We will refer to universal constants by $C \geq 1$ and $0 < c \leq 1$ and allow such constants to change from line to line of the computation.
- ▷ Denote $a \vee b = \max\{a, b)$ and $a \wedge b = \min(a, b)$, respectively, for the maximum and minimum of two real numbers a and b.
- ▷ We often abuse notation and write $\Omega(t)$ instead of $\Omega(u(t))$ etc.
- ▷ u^* and u_* denote the upper-semicontinuous envelope and the lower-semicontinuous envelope of u, respectively, see (4.4). We write USC and LSC respectively as shorthand for upper semicontinuous and lower semicontinuous.
- ▷ $F + H_0^1(U)$ refers to the space of functions in $H^1(U)$ with trace F on ∂U.

2. Motivating examples

In order to introduce the problem and motivate the phenomena we will study in the paper we present several examples with analytical computations and numerical simulations.

2.1. **Numerical simulations.** The obstacle solutions (O) can be relatively easily approximated by a large time limit ($\tau \to \infty$) of a dynamic contact angle problem: Suppose that F is increasing on $[s, t]$. Given $u(s)$, $u(t)$ can be found as the limit $\tau \to \infty$ of the unique, monotone solution of the free boundary problem

$$\begin{cases} -\Delta w(\tau) = 0, & \text{in } \{w(\tau) > 0\} \cap U, \\ V_n = \max(|\nabla w| - (1+\mu_+)^{1/2}, 0) & \text{on } \partial\{w(\tau) > 0\} \cap U, \\ w(\tau) = F(t), & \text{on } \partial U, \\ \{w(0) > 0\} = \{u(s) > 0\}. \end{cases}$$

Analogously, for F decreasing on $[s, t]$ we replace the velocity law by $V_n = \min(|\nabla w| - (1-\mu_-)^{1/2}, 0)$.

A numerical solution can be found by adapting the level set method introduced in [23], stopping at τ when $|V_n| < \varepsilon$ for some small parameter $\varepsilon > 0$. Plots in Figure 2 were produced this way.

2.2. **Jump conditions for (global) energetic vs obstacle solutions.** Next we consider an example where the jump time for the energy solutions is different from the jump time for the obstacle evolution. Heuristically speaking the obstacle evolution solutions jump as late and as little as possible, while the (global) energy solutions jump whenever it becomes energetically favorable.

Consider a domain U which is the complement of two disjoint closed disks and initial data given by two disjoint annuli. Then under increasing $F(t)$ the (O) solution will consist of two disjoint annuli until the value of the forcing when the boundaries of the two annuli meet at a single point. As F continues increasing past that critical value the solution will need to jump outwards to a new state. This situation is depicted in the simulations in Figure 2.

The energy solution with the same data and forcing must jump before the value of the forcing when the two annuli touch. One way to see this is that the blow-up at the touching point of the annuli is a two plane solution of the form

$$v(x) = (1 + \mu_+)^{1/2}|x \cdot e|.$$

However this blow-up does not satisfy the global stability condition in (E), since the harmonic replacement in any open region has the same positivity set but lower Dirichlet energy. Since any blow-up of a globally stable state is also globally stable we conclude that no energy solution can coincide with the obstacle solution all the way to the jump time.

2.3. **Convexity is not preserved under the obstacle evolution.** Consider the Bernoulli free boundary problem in the complement of a convex obstacle K

$$-\Delta u = 0 \ \text{ in } \ \{u > 0\} \setminus K, \ \text{ with } \ |\nabla u| = 1 \ \text{ on } \ \partial\{u > 0\} \setminus K.$$

It is known, see Henrot and Shahgholian [24], that the (unique) compactly supported solution of this problem has convex super-level sets. In particular if one considers the obstacle solution (O) without pinning $\mu_\pm = 0$, then the solution $u(t)$ of the obstacle evolution (which depends only on the current value $F(t)$ due to the lack of hysteresis) is convex at all times.

It is natural to ask whether convexity is still preserved under (O) in the case of nontrivial pinning interval $\mu_\pm > 0$. In fact it is not. We give a simulation of a counterexample in Figure 2 and present a sketched proof here.

Consider the case of $K = B_1$, $F(0) = 1$, and an initial region Ω_0 which is a "stadium" type initial data which is a large portion of the strip $-a < x_2 < a$ capped off by two circles of radius a centered at $(\pm b, 0)$. Here $0 < a - 1 \ll 1$ and $b \gg 1$. The initial region Ω_0 and K uniquely determine the initial profile u_0, which does have convex super-level sets due to [10]. Fix the pinning interval by the relations

$$\mu_+ := \max_{\partial\Omega_0}(|\nabla u_0|^2 - 1) \ \text{ and } \ \mu_- := \max_{\partial\Omega_0}(1 - |\nabla u_0|^2).$$

By symmetry considerations the first maximum is achieved at $(0, \pm a)$ and the second at $(\pm(b + a), 0)$. We can guarantee that both maxima are positive by choosing the a close to 1 and b large. Now increasing $F(t)$ slightly above 1 the free boundary needs to immediately move outwards near $(0, \pm a)$ since the slope there is already saturated at $|\nabla u_0(0, a)|^2 = 1 + \mu_+$, but since this is exactly the maximum value of the slope the outwards movement will only be in a small neighborhood of those two points. This motion must produce nonconvexity because the domain $\Omega(t)$ will have outward normal e_2 both at some point $(0, a + \delta(t))$ and at (b, a).

This argument could be justified rigorously using Theorem 1.5.

2.4. **Hysteresis.** In order to understand the macroscopic effect of the hysteresis inherent in the slope pinning condition we consider a simple explicitly solvable

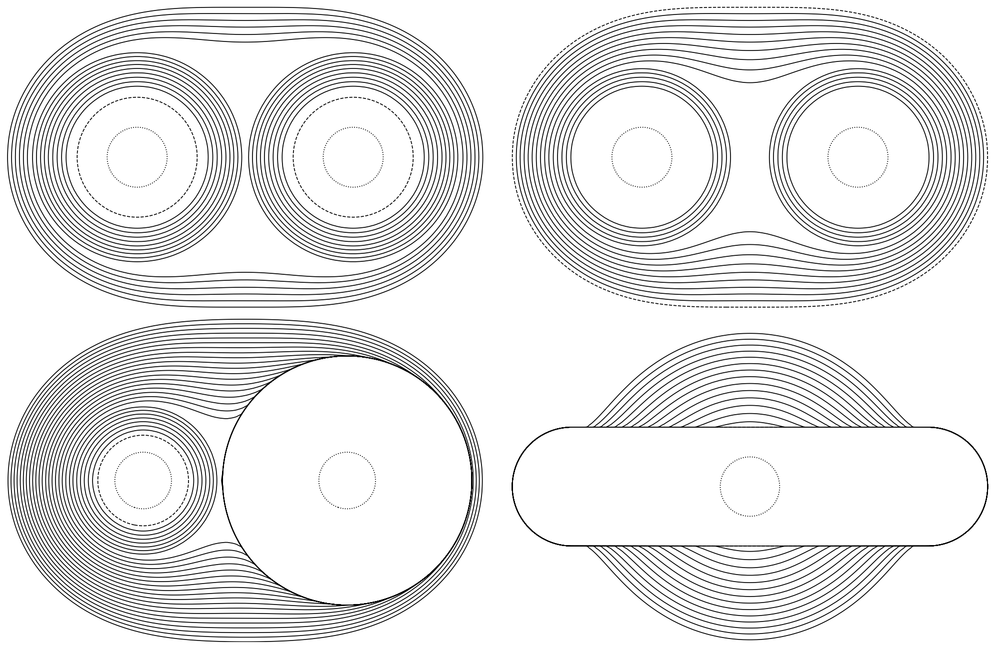

FIGURE 2. Plots of boundaries of obstacle solution (O) simulations. Solid curves represent $\partial\Omega(t) \cap U$ plotted for evenly spaced values of $F(t)$, with the initial shape dashed. ∂U is given by a dotted curve. Top left: Disconnected annuli initial data, jump discontinuity on touching. Top right: Receding situation (decreasing $F(t)$) with initial data given by the last step of the top left image. Note that the jump occurs at a different configuration, as late as possible. Bottom left: Different radius annuli, free boundary peels from the larger annulus after the jump. Bottom right: Stadium type initial data, convexity is not preserved.

radially symmetric example in dimension $d = 2$. See Figure 3 which will be further explained below.

Denote the family of radially symmetric solutions $V_{\lambda,F}$ of

$$\begin{cases} \Delta V_{\lambda,F} = 0 & \text{in } \ \Omega(V_{\lambda,F}) \setminus \overline{B_1}, \\ V_{\lambda,F} = F & \text{on } \ \partial B_1, \\ |\nabla V_{\lambda,F}| = \lambda & \text{on } \ \partial\Omega(V_{\lambda,F}) \setminus \overline{B_1}. \end{cases}$$

These can be explicitly computed as

$$V_{\lambda,F}(x) = F\left(1 - \frac{\log|x|}{\log\zeta(\lambda^{-1}F)}\right)_+$$

where $\zeta : (0,\infty) \to (1,\infty)$ strictly monotone increasing is the inverse of $R \mapsto R\log R$ i.e.

$$\zeta(s)\log\zeta(s) = s \quad \text{for} \quad s > 0.$$

Note that the radius of the support of $V_{\lambda,F}$ is $R = \zeta(\lambda^{-1}F)$.

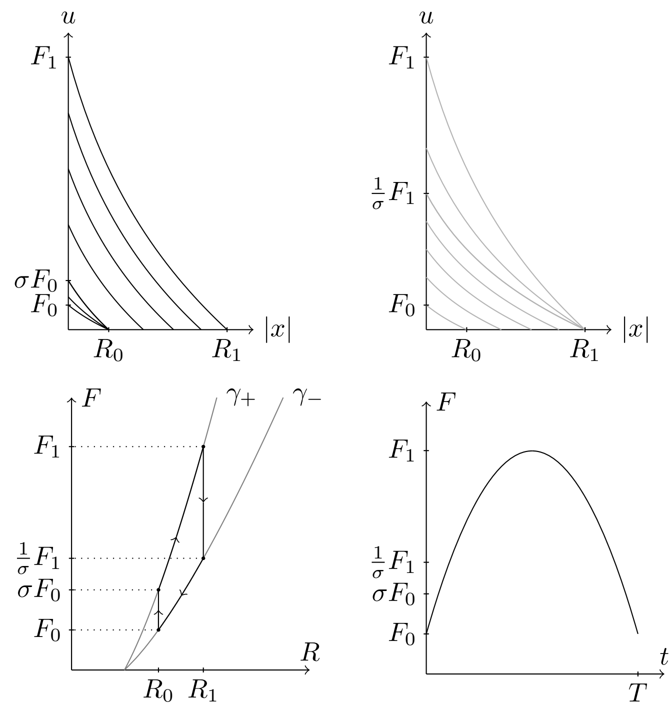

FIGURE 3. Hysteresis loop for radial solutions. Top left: profiles of advancing solution starting from $(R_0, F_0) \in \gamma_-$. Top right: profiles of receding solution starting at $(R_1, F_1) \in \gamma_+$. Bottom left: hysteresis loop diagram in (R, F)-plane. Bottom right: plot of $F(t)$. Parameters used to generate top and bottom set of pictures do not exactly match in order to better display the respective graphs. The factor σ is defined to be $\left(\frac{1+\mu_+}{1-\mu_-}\right)^{1/2}$.

Now let's consider the radially symmetric solution of (M) in $U = \mathbb{R}^2 \setminus B_1$ with some forcing $F(t)$ and

$$u_0 = V_{(1-\mu_-)^{1/2}, F_0}.$$

The solution must be of the form

$$u(t) = V_{\lambda(t), F(t)} \quad \text{with} \quad \lambda(t)^2 \in [1-\mu_-, 1+\mu_+]$$

and the dynamic slope condition becomes

$$\pm \frac{d}{dt} R(t) = \pm \frac{d}{dt} \zeta(\lambda(t)^{-1} F(t)) > 0 \quad \text{implies} \quad \lambda(t) = 1 \pm \mu_\pm.$$

In other words

$$\pm \frac{d}{dt} R(t) > 0 \quad \text{implies} \quad R(t) = \zeta((1 \pm \mu_\pm)^{1/2} F(t)).$$

or viewing matters in the (R, F) plane the state of the system is always in the region

$$\mathcal{S} = \left\{ (R, F) \in [1, \infty) \times (0, \infty) : \ \zeta((1+\mu_+)^{-1/2} F) \leq R \leq \zeta((1-\mu_-)^{-1/2} F) \right\}$$

and R can only increase/decrease while on the respective boundary curves

$$\gamma_\pm = \{(R, F) : \ R = \zeta((1 \pm \mu_\pm)^{-1} F)\}.$$

3. Regularity theory of Bernoulli obstacle problems

In this section we consider a pair of obstacle problems for the Bernoulli free boundary problem, one with an obstacle from above and the other with an obstacle from below. The two problems are similar but not exactly symmetric, as we will see below in the analysis. The regularity theory of the problem with obstacle from above has been developed by Chang-Lara and Savin [12], and as we were finishing preparing this paper their theory has been extended to the obstacle from below case by Ferreri and Velichkov [22]. We present the problems, recall the flat implies smooth regularity results from the above-mentioned works, and then show how to achieve the initial flatness and full regularity under a cone monotonicity hypothesis which is appropriate for our work.

3.1. **Bernoulli obstacle problems.** Let U be an open region, the domain. We say that u is a solution / supersolution / subsolution of the (unconstrained) Bernoulli free boundary problem in U if

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap U \\ |\nabla u| = 1 & \text{on } \ \partial\{u > 0\} \cap U. \end{cases} \tag{3.1}$$

Let the *obstacle* O be another open region with $C^{1,\alpha}$ boundary.

Definition 3.1. A function $u \in C(\overline{U})$ is a solution of the Bernoulli problem in U with obstacle O from below if

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap U \\ u > 0 & \text{in } O \cap U \\ |\nabla u| = 1 & \text{on } \ (\partial\{u > 0\} \setminus \overline{O}) \\ |\nabla u| \leq 1 & \text{on } \Lambda := \ (\partial\{u > 0\} \cap \partial O) \end{cases} \tag{3.2}$$

Definition 3.2. A function $u \in C(\overline{U})$ is a solution of the Bernoulli problem in U with obstacle O from above if

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap U \\ u = 0 & \text{in } \ \overline{U} \setminus O \\ |\nabla u| = 1 & \text{on } \ (\partial\{u > 0\} \cap O) \\ |\nabla u| \geq 1 & \text{on } \Lambda := \ (\partial\{u > 0\} \cap \partial O). \end{cases} \tag{3.3}$$

See Figure 4 for depictions of obstacle solutions.

3.2. **Additional hypotheses.** The obstacle below (resp. above) problems do not provide any a-priori bound on the slope at the free boundary from below (resp. above). Given a regular domain O and a specific boundary data on ∂U one could establish such bounds on the interior. However it is more convenient for us to just list these additional bounds as hypotheses which will be in force for some (but not all) of the statements below. Let $0 < \kappa < 1$ and consider the hypothesis: in the obstacle from below case

$$|\nabla u| \geq \kappa \ \text{ in the viscosity sense on } \ \partial\{u > 0\} \cap U, \tag{3.4}$$

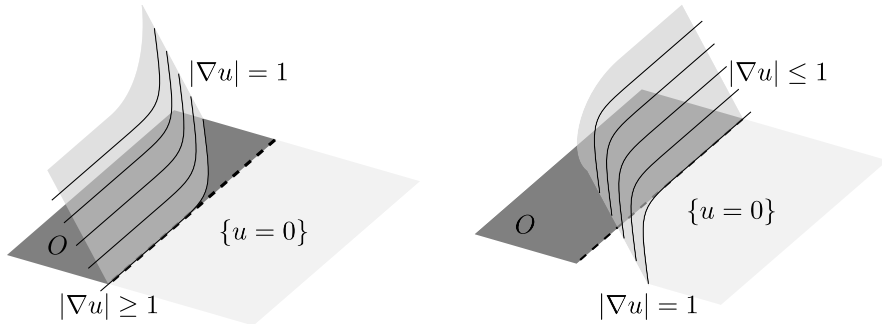

FIGURE 4. Left: Obstacle from above, slope is larger than 1 everywhere and saturates where free boundary bends into O. Right: Obstacle from below, slope is smaller than 1 everywhere and saturates where free boundary bends away from $\overline{O}$.

and in the obstacle from above case

$$|\nabla u| \leq \kappa^{-1} \text{ in the viscosity sense on } \partial\{u > 0\} \cap U. \tag{3.5}$$

3.3. **Existence and typical examples.** An indicative example of u solving a Bernoulli problem with obstacle from below (3.2) is the Perron's method obstacle minimal supersolution

$$u(x) := \inf\{v(x) : v \text{ is a supersolution of (3.1)}, \ v = g \text{ on } \partial U, \text{ and } v > 0 \text{ in } O \cap \overline{U}\}. \tag{3.6}$$

Here $g \in C(\partial U)$ is some boundary condition. In order for the minimal supersolution to also be positive on O we need to put some additional condition on g. Notice that every supersolution in (3.6) is above w, the solution of the Dirichlet problem

$$\Delta w = 0 \text{ in } O \cap U, \ w = 0 \text{ on } \partial O \cap U, \text{ and } w = g \text{ on } \partial U \cap \overline{O}.$$

So if $w > 0$ in O then u will be positive in O as well. In particular it would suffice to assume that

$$g > 0 \text{ on } O \cap \partial U.$$

The obstacle solution property follows from the typical Perron's method arguments. Local upward perturbations are possible everywhere in U, so u is a supersolution everywhere, while local downward perturbations are possible only away from $\partial\{u > 0\} \cap \overline{O}$ limiting the subsolution property.

An indicative example of u solving a Bernoulli problem with obstacle from above (3.3) is the obstacle maximal subsolution

$$u(x) := \sup\{v(x) : v \text{ a subsolution of (3.1) in } U, \ v = g \text{ on } \partial U, \text{ and } \{v > 0\} \subset O\}. \tag{3.7}$$

For consistency the positivity set of g on ∂U should be contained in $O \cap \partial U$. As before Perron's method applies, arbitrary downward perturbations are possible so u is a subsolution everywhere, but local upward perturbations are only allowed away from $\partial\{u > 0\} \cap \overline{O}$.

Remark 3.3. As in [12] it is also possible to construct solutions to (3.3) by minimal supersolution Perron's method, or by energy minimization. However, it is the maximal subsolutions that we will encounter in this work. This is something we need to be careful with, since non-degeneracy at the free boundary is a more delicate issue for maximal subsolutions.

3.4. **Flat implies $C^{1,\frac{1}{2}-}$ and regularity of cone monotone solutions.** First let us recall the flat implies $C^{1,1-}$ regularity away from the obstacle. This result is originally due to Caffarelli [7,8]. A more recent alternative proof by De Silva [15] has motivated the techniques used to study the obstacle problems we are considering..

Theorem 3.4. *(flat implies $C^{1,1-}$ for unconstrained Bernoulli [7,8]) Let u solves (3.1) in B_1 and $0 \in \partial\{u > 0\}$. For any $\beta \in (0,1)$ there is $\varepsilon_0 > 0$ and $C \geq 1$ universal so that if*

$$(x \cdot e - \varepsilon)_+ \leq u(x) \leq (x \cdot e + \varepsilon)_+ \quad in \quad B_1$$

for some $\varepsilon \leq \varepsilon_0$, then for all $0 < r < 1$ and some $|\nabla u(0)| = 1$

$$(x \cdot \nabla u(0) - C\varepsilon r^{1+\beta})_+ \leq u(x) \leq (x \cdot \nabla u(0) + C\varepsilon r^{1+\beta})_+ \quad in \quad B_r.$$

Next we recall a similar flat implies smooth result on the contact set between the solution and the obstacle. Consider the contact set of the free boundary with the obstacle $\Lambda := \partial\{u > 0\} \cap \partial O$. Let us denote $\partial'\Lambda$ to be the boundary of Λ relative to ∂O.

Theorem 3.5. *(flat implies $C^{1,\frac{1}{2}-}$ at the contact set [12, 22]) Let u solves either (3.3) or (3.2) in B_1 and $0 \in \partial'\Lambda$, O a $C^{1,\alpha}$ obstacle, and e the inward normal to O at 0. For any $0 < \beta < \min\{\alpha, \frac{1}{2}\}$ there is $\varepsilon_0 > 0$ and $C \geq 1$ universal so that if*

$$(x \cdot e - \varepsilon)_+ \leq u(x) \leq (x \cdot e + \varepsilon)_+ \quad in \quad B_1$$

for some $\varepsilon \leq \varepsilon_0$, then for all $0 < r < 1$

$$(x \cdot e - C\varepsilon r^{1+\beta})_+ \leq u(x) \leq (x \cdot e + C\varepsilon r^{1+\beta})_+ \quad in \quad B_r.$$

In both the obstacle above and obstacle below cases the proof is based on establishing an asymptotic expansion for flat solutions of the form

$$u(x) \sim (x \cdot e + \varepsilon w(x) + o(\varepsilon))_+$$

where w is a solution of a certain Signorini or thin obstacle problem and ε is the flatness in B_1 as in the hypothesis of Theorem 3.5. The $C^{1,\frac{1}{2}}$ optimal regularity for the Signorini problem is the reason for the $C^{1,\frac{1}{2}-}$ regularity which appears here, and is likely (almost) optimal. The idea of the argument, which is based on a compactness principle using a special Harnack inequality for flat solutions, goes back to the work of De Silva [15].

It is important for us to obtain a full regularity result without the flatness assumption. Of course some assumption is still necessary, singular solutions exist in higher dimensions, and fitting with the strongly star-shaped geometry we study later in the paper we will consider obstacle solutions satisfying a cone monotonicity condition. For the statement define

$$\mathrm{Cone}(e, \theta) := \{e' : e' \cdot e \geq 1 - \theta\} \quad \text{for} \quad e \in S^{d-1} \quad \text{and} \quad \theta \in (0, 1). \tag{3.8}$$

Theorem 3.6. *Suppose O is a bounded set with $C^{1,\alpha}$ boundary, and that u solves either (3.3) and (3.5) or (3.2) and (3.4) in B_1. Assume also that there are $e \in S^{d-1}$ and $1 \geq c_0 > 0$ so that u is monotone increasing in the directions of the cone* $\mathrm{Cone}(e, c_0)$.

Then for any $0 < \beta < \min\{\alpha, \frac{1}{2}\}$ the positivity set $\{u > 0\}$ is a $C^{1,\beta}$ domain in $B_{1/2}$ and $u \in C^{1,\beta}(\overline{\{u > 0\}} \cap B_{1/2})$. The $C^{1,\beta}$ norms depend only on α, κ, c_0, d, and the $C^{1,\alpha}$ norm of ∂O.

Remark 3.7. Chang-Lara and Savin [12] actually prove *optimal* $C^{1,\frac{1}{2}}$ regularity. This matches the regularity of the thin obstacle / Signorini problem, which appears as the first order term in the asymptotic expansion for ε-flat solutions in the flatness parameter ε. Since this optimal regularity requires significantly more work, and it is not necessary for our purposes here, we will not address that question for the obstacle from below problem.

We need to establish the initial flatness in order to apply Theorem 3.4 and/or Theorem 3.5 [12]. This is a bit different in our case from the way that initial flatness is established in [12] (minimal supersolutions) or [22] (one-sided energy minimizers).

The first difference is in the obstacle from below case. This issue appears in Lemma 3.8, where we establish the initial flatness of u at points of $\partial'\Lambda$ based on the existence of a non-tangential slope. In the case of the obstacle from above (3.3) the positive set of u lies inside of O which has smooth boundary. This means that a non-tangential gradient of u at the free boundary point in contact with ∂O can describe the leading-order behavior of u near the point in the entire positive set. This is no longer true with (3.2), where the positive set contains O. This motivates the cone monotonicity hypothesis on u in the obstacle from below case.

The other difference is in the obstacle from above case. Unlike [12], who consider minimal supersolutions and energy minimizers, we are interested to study maximal subsolutions. Non-degeneracy at the free boundary is not known in general for maximal subsolutions. However, in the case when the free boundary is a Lipschitz graph, in particular in the cone monotone case, non-degeneracy does hold for all viscosity solutions including the maximal subsolution, see Lemma A.1 below. This motivates the cone monotonicity hypothesis on u in the obstacle from above case.

In both cases the cone monotonicity hypothesis is probably overly strong, however it suffices for the applications in this paper. The possibility of a more general regularity result is an open question.

We also remark that Lipschitz regularity is easier than non-degeneracy and just follows from the viscosity supersolution property (3.5), see Lemma 5.5 below.

3.5. **Initial free boundary flatness.** In order to obtain the initial flatness we will show the blow-up limit at a free boundary point on the contact set Λ exists and is a half-planar supersolution of the unconstrained problem (3.1).

Lemma 3.8. *Assume that u is a solution of (3.2) and (3.4) (or (3.3) and (3.5)) in B_1 which is monotone with respect to the directions of a cone $\mathrm{Cone}(e, c_0)$, then*

$$\lim_{r\to 0} \frac{u(rx)}{r} = (\nabla u(0)\cdot x)_+ \quad \text{locally uniformly in } \mathbb{R}^d \tag{3.9}$$

with some gradient $c(d,\kappa) < |\nabla u(0)| \leq 1$ (resp. $1 \leq |\nabla u(0)| \leq C(d,\kappa)$). In particular, for any $\varepsilon > 0$ there exists $r_0 > 0$ sufficiently small (depending on u) so that for any $r \leq r_0$

$$|\nabla u|(0)(x_d - \varepsilon r)_+ \leq u(x) \leq |\nabla u|(0)(x_d + \varepsilon r)_+ \quad \text{in } B_r. \tag{3.10}$$

Proof. We will just consider the obstacle from below (3.2), the obstacle above case is in [12, Lemma 2.6]. The idea is that the interior ball condition furnished by the

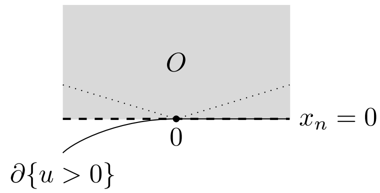

FIGURE 5. Asymptotic expansion in non-tangential cone plus monotonicity also gives control in $\{x_n \leq 0\}$.

regular obstacle gives a non-tangential blow-up limit and then the cone monotonicity upgrades this to a local uniform limit on the whole space.

Note that u is harmonic in its positivity set, $\{u > 0\} \supset O$ in B_1 and O is a half-space in B_1. Therefore 0 is an inner regular point so there is a slope $\nabla u(0)$, parallel to the inward normal e_d to O at 0, such that

$$u(x) = (\nabla u(0) \cdot x)_+ + o(|x|) \text{ as } x \to 0 \text{ non-tangentially in } O.$$

See [6, Lemma 11.17] for the proof, which does extend to the case of Hölder continuous coefficients. By non-degeneracy, Lemma A.1, and the Lipschitz estimate, standard recalled in Lemma 5.5 below, $|\nabla u(0)|$ is bounded from below away from zero, and from above.

Here is where we need the cone monotonicity property. The non-tangential information, by itself, does not give us any control in $B_1 \setminus O$.

From the non-tangential limit, for any $\varepsilon > 0$, there is $r_0 > 0$ such that

$$u(x) \leq 2|\nabla u(0)|\varepsilon r \ \text{ on } \ \{x_d = \varepsilon r\} \cap B_r \ \text{ for all } \ r \leq r_0.$$

Since u is monotone increasing in the e_d direction also

$$u(x) \leq 2|\nabla u(0)|\varepsilon r \ \text{ on } \ \{x_d \leq \varepsilon r\} \cap B_r \ \text{ for all } \ r \leq r_0.$$

See Figure 5.

Thus $\lim_{r\to 0} \frac{u(rx)}{r} = 0$ for $x_d \leq 0$ and we have upgraded the non-tangential limit to local uniform convergence of the blow-up sequence (3.9). Then the uniform stability of viscosity solutions implies that $|\nabla u(0)| \leq 1$.

Using non-degeneracy again we can prove the convergence of the free boundaries as well (3.10).

□

If we use Lemma 3.8 as stated then the initial flatness radius r_0 would depend on u is a non-universal way. In turn the $C^{1,\beta}$ norm of the solution depends on the initial flatness, and would also depend on u in a non-universal way. Next we show, using the flat implies smooth results, that r_0 is actually universal.

Lemma 3.9. *Assume that u is a solution of (3.2) and (3.4) (or (3.3) and (3.5)) in B_1 which is monotone with respect to the directions of a cone* $\mathrm{Cone}(e, c_0)$. *For any $\varepsilon > 0$ there exists $r_0 > 0$ sufficiently small, depending only on ε, c_0, κ, d and the $C^{1,\alpha}$ property of O, so that for any $r \leq r_0$ there is $\kappa \leq q \leq 1$ (resp. $1 \leq q \leq \kappa^{-1}$) so that*

$$q(x_d - \varepsilon r)_+ \leq u(x) \leq q(x_d + \varepsilon r)_+ \quad \text{in} \quad B_r. \tag{3.11}$$

Proof. We just consider the obstacle from below case, the obstacle from above case is similar. The proof is by compactness. Suppose otherwise, then there is $\varepsilon_0 > 0$, a sequence of u_k solving (3.2) in B_1, a sequence of obstacles O_k all uniformly $C^{1,\alpha}$ regular, and a sequence of radii $r_k \to 0$ so that (3.11) fails with $r = r_k$ and $\varepsilon = \varepsilon_0$. By uniform Lipschitz regularity and non-degeneracy we can assume that, on compact subsets of B_1, the u_k converge uniformly to some u, the $\partial\{u_k > 0\}$ converge in Hausdorff distance to $\partial\{u > 0\}$, and the O_k and ∂O_k also converge in Hausdorff distance to another $C^{1,\alpha}$ domain O. Standard viscosity solution arguments show that u solves (3.2) in B_1. This implies that u is in $C^{1,\beta}(\overline{\{u > 0\}} \cap B_{1/2})$ so for any $\varepsilon > 0$ there is $r_1 > 0$ so that (3.11) holds for $r \leq r_1$. Let $\varepsilon_1 > 0$ sufficiently small so that Theorem 3.4 and Theorem 3.5 hold, and then $r_1 > 0$ small so that

$$|\nabla u|(0)(x_d - \tfrac{1}{2}\varepsilon_1 r_1)_+ \leq u(x) \leq |\nabla u|(0)(x_d + \tfrac{1}{2}\varepsilon_1 r_1)_+ \ \text{ in } \ B_{r_1}$$

which implies, for k large enough,

$$|\nabla u|(0)(x_d - \varepsilon_1 r_1)_+ \leq u_k(x) \leq |\nabla u|(0)(x_d + \varepsilon_1 r_1)_+ \ \text{ in } \ B_{r_1}.$$

Then, applying either Theorem 3.4 or Theorem 3.5 as the case may be, implies that that for all $0 < r \leq r_1$

$$|\nabla u_k|(0)(x_d - C\varepsilon_1 r^{1+\beta})_+ \leq u(x) \leq |\nabla u_k|(0)(x_d + C\varepsilon_1 r^{1+\beta})_+ \ \text{ in } \ B_r$$

which contradicts the hypothesis for large k. □

4. Local laws: weak solutions, regularity, and comparison principle

In this section we study the viscosity solution notions (O) and (M), especially in the star-shaped setting. We establish regularity of (O) solutions using the results of Section 3. The central element of a viscosity solutions theory is always the comparison principle. One of the main results of this section is a comparison principle in the star-shaped setting, Proposition 4.26, between the (O) solution and an arbitrary (M) solution.

4.1. **Definitions and main results.** Let $F = F(t)$ be a strictly positive C^1 function on the time interval $[0, T]$ with a discrete set of critical points, or more precisely,

$$F \in C^1([0,T]) \quad \text{and} \quad Z := \{t \in (0,T) : F'(t) = 0\} \text{ is discrete.} \tag{4.1}$$

In other words, F only changes monotonicity at most finitely many times on any bounded interval.

We start by introducing a notion of solution that states explicitly what equations the obstacle solution (O) satisfies, see Lemma 4.18 below.

Definition 4.1. A function $u : [0,T) \times \overline{U} \to [0,\infty)$ is an *obstacle viscosity solution* (OVS) if $u(t)$ is continuous for each $t \in [0,T]$, satisfies $u(t) = F(t)$ on ∂U, and the following holds for any $(s,t) \subset (0,T) \setminus Z$:

(a) If F is monotone increasing on $[s,t]$ then $u(t)$ solves in the viscosity sense

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega(u(t)) \cap U \\ u(t) > 0 & \text{in } \ \Omega(u(s)) \cap U \\ |\nabla u(t)|^2 \leq 1 + \mu_+ & \text{on } \ (\partial\Omega(u(t)) \cap \partial\Omega(u(s))) \cap U \\ |\nabla u(t)|^2 = 1 + \mu_+ & \text{on } \ (\partial\Omega(u(t)) \setminus \overline{\Omega(u(s))}) \cap U \end{cases}$$

(b) If F is monotone decreasing on $[s,t]$ then $u(t)$ solves in the viscosity sense

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega(u(t)) \cap U \\ u(t) = 0 & \text{in } \overline{U} \setminus \Omega(u(s)) \\ |\nabla u(t)|^2 = 1 - \mu_- & \text{on } (\partial\Omega(u(t)) \cap \Omega(u(s))) \cap U \\ |\nabla u(t)|^2 \geq 1 - \mu_- & \text{on } (\partial\Omega(u(t)) \cap \partial\Omega(u(s))) \cap U. \end{cases}$$

Remark 4.2. Note in particular in case of (a) u is monotone increasing on $[s,t]$, and in case of (b) u is monotone decreasing in $[s,t]$. Given the condition on F, it follows that u also only changes monotonicity finitely many times on any bounded interval.

4.1.1. *Slope condition in the comparison sense.* The slope conditions are interpreted in the comparison / viscosity sense which we make precise in two ways. First we define a notion of sub and superdifferential which is suited to the problem. This is also called the first order semi-jet $\mathcal{J}^{+,1}_{\{u>0\}}$ in [13].

Definition 4.3. Given a non-negative continuous function u on U define, for each $x_0 \in \partial\Omega(u) \cap U$

$$D_+u(x_0) = \Big\{p \in \mathbb{R}^d : u(x) \leq p \cdot (x - x_0) + o(|x - x_0|) \text{ in } \overline{\Omega(u)}\Big\}$$

Definition 4.4. Given a non-negative continuous function u on U define, for each $x_0 \in \partial\Omega(u) \cap U$

$$D_-u(x_0) = \big\{p \in \mathbb{R}^d : u(x) \geq p \cdot (x - x_0) - o(|x - x_0|)\big\}$$

This leads to a comparison notion of slope conditions.

Definition 4.5. Suppose $u : U \to [0,\infty)$ is continuous and is subharmonic in $\Omega(u) \cap U$. Then u is a subsolution of

$$|\nabla u|^2 \geq Q \quad \text{on} \quad \partial\Omega(u) \cap U$$

if, for every $x \in \partial\Omega(u)$ and every $p \in D_+u(x)$,

$$|p|^2 \geq Q.$$

Similarly, if $u : U \to [0,\infty)$ is continuous and is superharmonic in $\Omega(u) \cap U$, then u is a supersolution of

$$|\nabla u|^2 \leq Q \quad \text{on} \quad \partial\Omega(u) \cap U$$

if, for every $x \in \partial\Omega(u)$ and every $p \in D_-u(x)$,

$$|p|^2 \leq Q.$$

In this section we will define the spatial slope conditions in terms of the sub and superdifferential notions above. However, in later sections we prefer to use the usual touching test functions notion which is more convenient to argue with in many cases.

Definition 4.6. Suppose $u : U \to [0,\infty)$ is continuous and is subharmonic in $\Omega(u) \cap U$. Then u is a subsolution of

$$|\nabla u|^2 \geq Q \quad \text{on} \quad \partial\Omega(u) \cap U$$

if, whenever φ is a smooth test function such that φ_+ touches u from above at $x \in \partial\Omega(u) \cap U$ with $\Delta\varphi(x) < 0$,

$$|\nabla\varphi(x)|^2 \geq Q.$$

Similarly, suppose $u : U \to [0, \infty)$ is continuous and is superharmonic in $\Omega(u) \cap U$. Then u is a supersolution of

$$|\nabla u|^2 \leq Q \quad \text{on} \quad \partial\Omega(u) \cap U$$

if, whenever φ is a smooth test function touching u from above at $x \in \partial\Omega(u) \cap U$ with $\Delta\varphi(x) > 0$,

$$|\nabla\varphi(x)|^2 \leq Q.$$

These two notions of viscosity solution are equivalent.

Lemma 4.7. *Suppose that $u : U \to [0, \infty)$ is continuous and subharmonic (resp. superharmonic) in $\Omega(u) \cap U$. Then u is a viscosity subsolution (resp. supersolution) in the sense of Definition 4.5 if and only if it is a viscosity subsolution (resp. supersolution) in the sense of Definition 4.6.*

We provide a proof sketch in the appendix for convenience, see Section A.4.

Remark 4.8. Note that this equivalence requires both solution properties to hold *everywhere.* If φ touches u from above at $x_0 \in \partial\Omega(u) \cap U$ then $\nabla\varphi(x_0) \in D_+u(x_0)$. However the reverse is not true, if $p \in D_+u(x_0)$ it does not necessarily guarantee that u can be touched from above by a smooth test function at the *same point* with $\nabla\varphi(x_0) \approx p$, rather one can find a nearby point where the touching occurs.

In Section 7 this distinction is important since we only prove a sub/superdifferential notion $\mathcal{H}^{d-1}$-almost everywhere.

4.1.2. *Dynamic slope condition in the comparison sense.* In order to formulate the dynamic slope condition (1.5) we need a weak pointwise sense of positive and negative normal velocity. We will use a notion based on space-time "light cones" in place of standard barrier functions, see Figure 6 for a sketch of the geometry. This notion of viscosity solutions is stronger than the usual comparison definition of level set velocity, since we test more free boundary points that have weaker space-time regularity. Nonetheless, this stronger notion of normal velocity fits well with other notion of solutions that we use.

Definition 4.9. Given a time varying family of domains $\Omega(t)$ such that $\chi_{\Omega(t)}$ is upper semi-continuous, and for given $c, r_0 > 0$, we say that the *outward normal velocity* at $x_0 \in \partial\Omega(t_0)$ is at least c at scale r_0 and write

$$V^{r_0}(t_0, x_0) \geq c$$

if the *positive cone condition* holds, namely

$$\{x : \ |x - x_0| \leq c(t_0 - t)\} \subset \Omega(t)^{\complement} \quad \text{for} \quad t_0 > t \geq t_0 - r_0. \tag{4.2}$$

Define

$$V^+(t_0, x_0) := 0 \vee \sup\{c : \exists r_0 > 0 \ \text{ so that } \ V^{r_0}(t_0, x_0) \geq c\}.$$

We write $V(t_0, x_0) > 0$ if $V^+(t_0, x_0) > 0$, i.e., there are $c > 0$ and $r_0 > 0$ so that $V^{r_0}(t_0, x_0) \geq c$. If we need to clearly specify the domain we write $V(\cdot; \Omega)$, $V^{r_0}(\cdot; \Omega)$, etc.

Definition 4.10. Given a time varying family of domains $\Omega(t)$ such that $\chi_{\Omega(t)}$ is lower semi-continuous, and for given $c, r_0 > 0$, we say that the *inward normal velocity* at $x_0 \in \partial\Omega(t_0)$ is at least c at scale r_0 and write

$$V^{r_0}(t_0, x_0) \leq -c$$

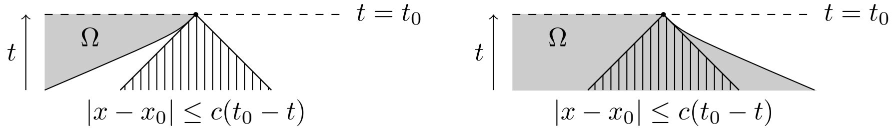

FIGURE 6. Left: velocity c cone touches $\Omega(t)$ from the outside at (t_0, x_0), interpreted as $V_n(t_0, x_0) \ge c$. Right: velocity c cone touches $\Omega(t)$ from the inside at (t_0, x_0), interpreted as $V_n(t_0, x_0) \le -c$.

if the *negative cone condition* holds, namely

$$\{x : \ |x - x_0| \le c(t_0 - t)\} \subset \Omega(t) \quad \text{for} \quad t_0 > t \ge t_0 - r_0. \tag{4.3}$$

Define

$$V^-(t_0, x_0) := 0 \wedge \inf\{-c < 0 : \exists r_0 > 0 \ \text{ so that } \ V^{r_0}(t_0, x_0) \le -c\}.$$

We write $V(t_0, x_0) < 0$ if $V^-(t_0, x_0) < 0$, i.e., there are $c > 0$ and $r_0 > 0$ so that $V^{r_0}(t_0, x_0) \le -c$. If we need to clearly specify the domain we write $V(\cdot; \Omega)$, $V^{r_0}(\cdot; \Omega)$, etc.

Finally, let us recall the semicontinuous envelopes u^* and u_* of a function $u : [0, T] \times \overline{U} \to \mathbb{R}$:

$$u^*(t, x) := \limsup_{(s,y)\to(t,x)} u(s, y), \qquad u_*(t, x) := \liminf_{(s,y)\to(t,x)} u(s, y), \tag{4.4}$$

where (s, y) is always assumed to be in $[0, T] \times \overline{U}$. Recall that u^* is USC and u_* is LSC.

Equipped with the notions of nonzero normal velocity and subdifferentials we can now precisely define the meaning of the local stability condition and the dynamic slope condition of (M) in Definition 1.4:

Definition 4.11. We say that a bounded map $u : [0, T] \to C(\overline{U})$ is a *viscosity solution of* (1.4) *on* $[0, T] \times U$ *in the semicontinuous envelope sense* if

(a) $u(t)$ satisfies (1.4) for every $t \in [0, T]$.
(b) If $p \in D_+u^*(t_0, x_0)$ for some $x_0 \in \partial\Omega(u^*(t_0))$ then $|p|^2 \ge 1 - \mu_-$.

Remark 4.12. Note that (1.4) and uniform boundedness imply that $u(t)$ are uniformly Lipschitz continuous in compact subsets of U by Lemma 5.5. Therefore the upper and lower envelopes $u^*(t)$ and $u_*(t)$ are continuous in x for each t. It is also standard that $u^*(t)$ and $u_*(t)$ are respectively subharmonic and superharmonic in their positivity sets.

Remark 4.13. Note that if $u : [0, T] \to C(\overline{U})$ such that $u(t)$ is non-degenerate uniformly in t then (b) in Definition 4.11 follow from (a). Indeed, say that $u^*(t) - \phi$ has a strict local maximum at x_0 in $\overline{\Omega(u^*(t))}$. By the uniform nondegeneracy we can deduce that there exists a sequence $t_n \to t$ and $x_n \to x_0$ such that $u(t_n) - \phi$ has a local maximum at x_n in $\overline{\Omega(u(t_n))}$, from which it follows that $u^*(t)$ satisfies (1.4).

Remark 4.14. A similar argument to Remark 4.13 can be done for u_* but non-degeneracy is not necessary because the test functions touch from below in $\mathbb{R}^d$

instead of in $\Omega(u(t))$. So (a) actually directly implies the supersolution analogue to (b): if $p \in D_-u_*(t_0, x_0)$ for some $x_0 \in \partial\Omega(u_*(t_0))$, then $|p|^2 \le 1 + \mu_+$.

Definition 4.15. We say that $u : [0, T] \to C(\overline{U})$ is a *viscosity solution of (1.5) on* $[0, T] \times U$ *in the semicontinuous envelope sense* if

(a) If $V(t_0, x_0; \Omega(u^*)) > 0$ for some $x_0 \in \partial\Omega(u^*(t_0))$, then any $p \in D_+u^*(t_0, x_0)$ satisfies $|p|^2 \ge 1 + \mu_+$.
(b) If $V(t_0, x_0; \Omega(u_*)) < 0$ for some $x_0 \in \partial\Omega(u_*(t_0))$, then any $p \in D_-u_*(t_0, x_0)$ satisfies $|p|^2 \le 1 - \mu_-$. in $U \times (0, T]$.

4.1.3. *Star-shaped comparison principle / equivalence of solution notions.* We will show the equivalence of the notions in a strongly star-shaped setting.

Definition 4.16. In the following we say that a set is *strongly star-shaped* if it is star-shaped with respect to all x in a neighborhood of the origin. A function $u : U \supset \mathbb{R}^d \to \mathbb{R}$ is strongly star-shaped if each of its superlevel sets $\{u > \eta\} \cup U^{\complement}$ are strongly star-shaped.

The important property is that a strongly star-shaped set has a Lipschitz boundary. This can be seen by drawing the interior and exterior cones created by drawing the rays from every star-shaped center though a point on the boundary, see for example [20, Figure 2].

The main result of this section is the following theorem that asserts that in the strongly star-shaped setting all three notions of solutions that we introduced turn out to be equivalent.

Theorem 4.17. *If $\mathbb{R}^d \setminus U$ and Ω_0 are both bounded strongly star-shaped sets, Ω_0 is a $C^{1,\alpha}$ domain for some $\alpha > 0$, F satisfies* (4.1) *and $u(0) \in C(\overline{U})$ satisfies* (1.4)*, then*

$$u \text{ is (O) solution } \Leftrightarrow u \text{ is (OVS) solution} \Leftrightarrow u \text{ is (M) solution.}$$

Furthermore $\Omega(u(t))$ are $C^{1,\beta}$ domains uniformly in $t \in [0, T]$ for some $\beta > 0$.

Proof. To prove this theorem, we first show the easy equivalence of (O) and (OVS) that follows from a standard viscosity solution argument and the uniqueness of an obstacle Bernoulli problem in the star-shaped setting (Lemma 4.18). The fact that an obstacle solution (O) is also motion-law viscosity solution (M) is shown in Lemma 4.20. Then we deduce the regularity in both space and time of (O)/(OVS) solutions (Lemma 4.22, Corollary 4.25). Finally showing that (M) implies the other two notions (Corollary 4.27) follows a comparison-type viscosity argument using sup- and inf-convolutions between a regular (M) solution and a general (M) solution (Proposition 4.26). □

Lemma 4.18. *The obstacle solution* (O) *is also an obstacle viscosity solution* (OVS)*. Under the strong star-shapedness assumption of Theorem 4.17 this is the unique* (OVS) *solution.*

Proof. (O) ⇒ (OVS) can be verified by a standard viscosity solution argument and does not require star-shapedness.

(OVS) ⇒ (O) on the other hand follows from the uniqueness of the obstacle Bernoulli problem in the strongly star-shaped case: (O) always exists and by the previous step, it is also (OVS), and hence by Theorem A.4 it must be the unique (OVS). □

Remark 4.19. Let us point out that, while most of the analysis in this section regards star-shaped initial data, for general initial data we still have a unique (O) solution by its definition, and this solution is also an (OVS) by the lemma above. We do not know whether (OVS) would imply (M), or vice versa, for general initial data. However formally both notions (M) and (OVS) come with the same dynamic slope condition at space-time regular points of the free boundary.

4.2. **Obstacle solutions are motion law solutions.** Next we show that, in a general setting without star-shaped assumption, obstacle solutions of (O) solve the motion law (M).

Lemma 4.20. *If $u : [0,T] \to C(\overline{U})$ is an obstacle solution (O) on $[0,T]$ that is uniformly non-degenerate at its free boundary, then u solves (M).*

Proof. The state $u(t)$ is harmonic in $\Omega(u(t))$ by definition. By Remark 4.12 u^* and u_* are respectively sub and superharmonic in their positivity sets.

Now we aim to check the stability conditions in Definition 4.11(a). We first check the lower bound condition $|\nabla u(t)|^2 \geq 1 - \mu_-$ in the viscosity sense on $\partial\Omega(t) \cap U$ by induction on the monotonicity intervals of F provided by assumption (4.1). We know that the condition is satisfied at $t = 0$ by the compatibility of the initial data. Suppose now that we know that u satisfies this condition up to $s \in Z \cup \{0\}$ and consider $t > s$ such that $(s,t) \cap Z = \emptyset$.

- If F is decreasing on $[s,t]$ then the lower bound on $|\nabla u(t)|^2$ follows from (OVS)-(b) immediately.
- Otherwise F is increasing on $[s,t]$. In this case $u(t)$ are minimal supersolutions above $u(s)$ for $t_0 \in (s,t]$ and so, by Lemma A.1, they are uniformly non-degenerate. On $(\partial\Omega(t) \setminus \overline{\Omega(s)}) \cap U$ we get the same bound from (OVS)-(a) since $|\nabla u(t)|^2 = 1 + \mu_+ \geq 1 - \mu_-$. Finally, on $(\partial\Omega(t) \cap \partial\Omega(s)) \cap U$, as $u(t) \geq u(s)$, any test function touching $u(t)$ from above in $\overline{\{u(t) > 0\}}$ at $x \in (\partial\Omega(t) \cap \partial\Omega(s)) \cap U$ also touches $u(s)$ and hence $D_+u(t,x) \subset D_+u(s,x)$. Hence $u(t)$ is a solution $|\nabla u(t)|^2 \geq 1 - \mu_-$ at x in the viscosity sense since $u(s)$ is by the induction hypothesis.

By induction we can therefore extend the property to all $t \in [0,T]$ since Z is discrete. An analogous argument gives $|\nabla u(t)|^2 \leq 1 + \mu_+$.

By the assumption of uniform non-degeneracy, Remark 4.13 yields Definition 4.11(b). We note that the assumption is actually needed only when F is decreasing. When F is increasing, $u(t)$ is a minimal supersolution above $u(s)$ and hence non-degenerate by Lemma A.1.

We now check the remaining dynamic slope conditions. First consider monotone increasing interval $[s,t]$ of F and we aim to check the advancing condition Definition 4.15 part (a). Suppose the positive cone condition (4.2) holds for $x_0 \in \partial\Omega(u^*(t_0)) \cap U$ for some $s < t_0 \leq t$. By the positive cone condition and monotonicity there is a neighborhood $B_r(x_0) \subset \overline{\Omega(u(s))}^{\mathsf{C}}$, and thus we have $|\nabla u|^2 \geq 1 + \mu_+$ on $\partial\Omega(u)$ in $(s,t] \times B_r(x_0)$ by (OVS)-(a). If $t_0 = t$ then $u^*(t) = u(t)$ because $u(t)$ is a local maximum by increasing monotonicity to the left and decreasing monotonicity to the right, so we can reduce to $t_0 \in (s,t)$. Then, by uniform non-degeneracy of minimal supersolutions Lemma A.1 and Remark 4.13, we can conclude that $|\nabla u^*|^2 \geq 1 + \mu_+$ on $\partial\Omega(u^*)$ in $(s,t) \times B_r(x_0)$, in particular it also holds at (t_0, x_0).

Next we check that the receding condition Definition 4.15 part (b) holds for monotone decreasing interval $[s,t]$ of F. The argument is mostly the same with a subtle difference which we need to point out. Suppose the negative cone condition (4.3) holds for $x_0 \in \partial\Omega(u_*(t_0)) \cap U$ for some $s < t_0 \le t$. As before there is a neighborhood $B_r(x_0) \subset \Omega(u(s))$, and thus we have $|\nabla u|^2 \le 1 - \mu_-$ on $\partial\Omega(u)$ in $(s,t] \times B_r(x_0)$ by (OVS)-(b). If $t_0 = t$ then $u_*(t) = u(t)$ so we reduce to the case $t_0 \in (s,t)$. Then by Remark 4.14, we can conclude that $|\nabla u_*| \le 1 - \mu_-$ on $\partial\Omega(u^*)$ in $(s,t] \times B_r(x_0)$, in particular it also holds at (t_0, x_0). Notice that we do *not* need uniform non-degeneracy, which is not known in general for maximal subsolutions, because we are testing a supersolution condition. □

4.3. **Spatial regularity.** We start with star-shapedness of (O).

Lemma 4.21. *Suppose that u solves (O), $K = \mathbb{R}^d \setminus U$ and $\Omega_0 = \{u_0 > 0\}$ are strongly star-shaped, u_0 satisfies (1.4) and F satisfies (4.1). Then $u(t)$ is strongly star-shaped for every $t \ge 0$ with respect to the same set of centers as K and Ω_0.*

Proof. We show the star-shapedness only with respect to the origin assuming that K and Ω_0 are star-shaped with respect to the origin. The proof with respect to other centers can be done analogously by translation.

We will use induction on the monotonicity intervals of F. Suppose that $t > 0$ with $(0,t) \cap Z \ne \emptyset$. Consider the case $F(t) > F(0)$. First note that for $\lambda < 1$

$$v(x) = \begin{cases} \min\big(u(t,\lambda x), F(0)\big) & x \in \lambda^{-1}U, \\ F(0) & x \in U \setminus \lambda^{-1}U, \end{cases}$$

is a supersolution of (1.4). Since $\{u(0) > 0\}$ is star-shaped and $\{u(t) > 0\} \supset \{u(0) > 0\}$ also

$$\lambda^{-1}\{u(t) > 0\} \supset \{u(0) > 0\}.$$

Thus v is a supersolution of (1.4) above $u(0)$ and so

$$u(t, \lambda \cdot) \ge u(t) \qquad \text{in } \lambda^{-1}U$$

by the minimality of $u(t)$ in the definition of (O). Since $\lambda > 0$ was arbitrary, $u(t)$ is star-shaped (as are all of its super-level sets) with respect to 0.

In the case $F(t) < F(0)$ we consider the dilation with $\lambda > 1$ and easily check that $u(t,\lambda\cdot) \le u(0)$ in U and $u(t,\lambda\cdot)$ is a subsolution of (1.4) and hence $u(t,\lambda\cdot) \le u(t)$ in U by the maximality of $u(t)$ in the definition of (O). In particular, $u(t)$ is star-shaped with respect to 0.

By translation we have star-shapedness at any point at which K and $\{u_0 > 0\}$ are star-shaped. Finally by induction on the monotonicity intervals of F, we can continue the star-shapedness property up to any $t > 0$. □

A solution of (O) is a solution of the Bernoulli obstacle problem on each monotonicity interval of F, where the obstacle is the solution at the beginning of the interval. This allows us to use the regularity of the obstacle problem, Section 3, only a finite number of times to establish the regularity of (O) at an arbitrary time.

Lemma 4.22. *Suppose that u solves (O) with F satisfying (4.1), $K = \mathbb{R}^d \setminus U$ has C^2 boundary, Ω_0 has a $C^{1,\alpha}$ boundary, and both K and Ω_0 are strongly star-shaped. Then for any $0 < \beta < \min\{\alpha, \frac{1}{2}\}$ the domains $\{u(t) > 0\}$ are $C^{1,\beta}$ and $u(t) \in C^{1,\beta}(\overline{\{u(t) > 0\} \cap U})$ uniformly in $t \in [0,T]$.*

Proof. Due to the regularity of $K = \mathbb{R}^d \setminus U$ the function $u(t)$ is in $C^2(\{u(t) > 0\} \cap \partial U)$. We need to show the regularity at the free boundary of $u(t)$.

By Lemma 4.21 the sets $\{u(t) > 0\}$ are uniformly strongly star-shaped, and therefore they are uniformly Lipschitz in $[0, T]$. Let $t > 0$ be such that $(0, t) \cap Z = \emptyset$. Therefore by the regularity of the Bernoulli obstacle problem Theorem 3.6 with $C^{1,\alpha}$ obstacle $\{u_0 > 0\}$ and any $0 < \beta_0 < \min\{\alpha, \frac{1}{2}\}$ the domain $\{u(t) > 0\}$ is C^{1,β_0} and $u(t) \in C^{1,\beta_0}(\overline{\{u(t) > 0\} \cap U})$.

Consider now $t > 0$ such that $(0, t) \cap Z = \{s_1\}$. By the previous step $\{u(s_1) > 0\} \in C^{1,\beta_0}$. Applying the regularity of the Bernoulli obstacle problem with $\alpha = \beta_0$ for any $0 < \beta_1 < \beta_0$ we find that $u(t)$ has regularity C^{1,β_1}.

We can continue inductively on the monotonicity intervals of F up to $t = T$, just taking a strictly decreasing sequence of $\min\{\alpha, \frac{1}{2}\} > \beta_0 > \beta_1 > \ldots > \beta_{\#(0,T)\cap Z} = \beta$. ☐

4.4. **Time regularity.** The regularity in space Lemma 4.22 of a star-shaped (O) solution yields in particular that $u(t)$ is Lipschitz locally uniformly in time. This in particular implies (with the nondegeneracy of the solutions of the Bernoulli problem) that its support moves in a Lipschitz way as well.

Lemma 4.23. *Suppose that u solves (O), $K = \mathbb{R}^d \setminus U$ has C^2 boundary, and K and $\Omega_0 = \{u_0 > 0\}$ are strongly star-shaped and bounded, and u_0 satisfies (1.4). Then*

$$|u(t,x) - u(s,x)| \le C\|F'\|_\infty |t-s| \quad \text{in} \;\; U \tag{4.5}$$

where $C = C(L, R, \min F)$ with L the Lipschitz constant of u in space and $R := \max_{x\in K\cup\Omega_0} |x|$.

Proof. Step 1. By the minimality of $u(t)$, including $u(0)$ by (1.4),

$$u(t,x) \le \Big(F(t) + (1+\mu_+)(R - |x|)\Big)_+, \qquad t \in [0,T], x \in \overline{U},$$

since the right-hand side is a supersolution. Therefore

$$x \in \Omega(t) \quad \Rightarrow \quad |x| \le \frac{F(t)}{1+\mu_+} + R. \tag{4.6}$$

Step 2. Fix $(s,t) \subset \mathbb{R} \setminus Z$ with $F(s) < F(t)$. Let $\lambda := F(s)/F(t) < 1$. Define

$$v(x) := \begin{cases} \lambda^{-1} u(s, \lambda x), & x \in \lambda^{-1}\overline{U} \\ F(t), & \text{otherwise.} \end{cases}$$

By the choice of λ, v is continuous and hence a supersolution on U with boundary data $F(t)$. By minimality, $u(t) \le v$.

Fix $\lambda x \in \Omega(s) \cap U$. (4.6) implies $|x| \le \lambda^{-1}(\frac{F(s)}{1+\mu_+} + R)$. Recall also that $u(s, \lambda x) \le F(s)$. We have

$$\begin{aligned} v(x) = \lambda^{-1} u(s,\lambda x) &= u(s,\lambda x) + (\lambda^{-1} - 1) u(s, \lambda x) \\ &\le u(s,x) + \|\nabla u(s)\|_\infty (1-\lambda)|x| + (\lambda^{-1} - 1) F(s) \\ &\le u(s,x) + (\lambda^{-1} - 1)\left(L(\frac{F(s)}{1+\mu_+} + R) + F(s)\right). \end{aligned}$$

Since $u_t \le v$ we conclude that

$$0 \le u(t,x) - u(s,x) \le \left(\frac{L}{1+\mu_+} + 1 + \frac{R}{F(s)}\right)(F(t) - F(s)).$$

For $x \in U \setminus \lambda^{-1}U$ we note that $\operatorname{dist}(x, \partial U) \leq (\lambda^{-1} - 1)R = \frac{R}{F(s)}(F(t) - F(s))$ and therefore

$$\begin{aligned} 0 \leq u(t,x) - u(s,x) &\leq F(t) - F(s) + \|\nabla u(s)\|_\infty \operatorname{dist}(x, \partial U) \\ &\leq \left(1 + \frac{LR}{F(s)}\right)(F(t) - F(s)). \end{aligned}$$

Step 3. If $(s,t) \subset \mathbb{R} \setminus Z$ with $F(s) > F(t)$, we have $\lambda = F(s)/F(t) > 1$ and $v \leq F(t)$. Hence v is a subsolution and therefore by maximality $u(t) \geq v$. We only need to consider $x \in \Omega(s) \cap U$. This time

$$\begin{aligned} v(x) = \lambda^{-1}u(s, \lambda x) &= u(s, \lambda x) + (\lambda^{-1} - 1)u(s, \lambda x) \\ &\geq u(s,x) - \|\nabla u(s)\|_\infty (\lambda - 1)|x| - (1 - \lambda^{-1})F(s) \\ &\leq u(s,x) - \left(L(\frac{F(s)}{1 + \mu_+} + R\right)\frac{F(s) - F(t)}{F(t)} - F(t) + F(s). \end{aligned}$$

Hence by $u(t) \geq v$

$$0 \geq u(t,x) - u(s,x) \geq L\left(\frac{F(s)}{(1 + \mu_+)F(t)} + 1 + \frac{R}{F(t)}\right)(F(s) - F(t)).$$

For general (s,t), we recover the estimate (4.5) by triangle inequality. $\square$

Using the nondegeneracy of cone monotone solutions of the Bernoulli problem, Lemma A.1, we can deduce Lipschitz regularity of $\Omega(u(t))$ in time.

Corollary 4.24. *Under the assumptions of Lemma 4.23, the sets $\Omega(u(t))$ are locally Lipschitz in time with respect to the Hausdorff distance.*

Proof. Let us fix $s \neq t$. Let $z \in \partial\Omega(u(t))$ be the point that maximizes the distance from $\partial\Omega(u(s))$. Let us denote $r := \operatorname{dist}(z, \partial\Omega(u(s)))$.

Suppose that $z \notin \Omega(u(s))$. We have $B_r(z) \cap \Omega(u(s)) = \emptyset$ and hence $u_s = 0$ on $B_r(z)$. On the other hand, the free boundary is locally a Lipschitz graph and so the non-degeneracy (note that $u(t)$ satisfies (1.4) by Lemma 4.20) for the Bernoulli problem in this setting Lemma A.1 yields $\sup_{B_r(z)} u(t) \geq c_0 r$. By the Lipschitz continuity of u in time, we deduce

$$c_0 r \leq \sup_{B_r(z)} u(t) \leq \tilde{L}|t - s|,$$

where $\tilde{L}$ is the Lipschitz constant in (4.5). In particular, $r \leq \frac{\tilde{L}}{c_0}|t - s|$.

If $z \in \Omega(u(s))$ then $B_r(z) \subset \Omega(u(s))$. Using the non-degeneracy, Harnack inequality and the Lipschitz regularity, we have

$$C_H c_0 \frac{r}{2} \leq C_H \sup_{B_{r/2}(z)} u(s) \leq u(s,z) \leq \tilde{L}|t - s|,$$

which yields $r \leq \frac{2\tilde{L}}{C_H c_0}|t - s|$. $\square$

Finally we mention that the time regularity and space regularity can be interpolated to get time regularity of ∇u.

Corollary 4.25. *Under the assumptions of Lemma 4.22 for any $0 < \beta < \min\{\alpha, \frac{1}{2}\}$*

$$u \in C_t^{0,1} C_x \cap L_t^\infty C_x^{1,\beta} \quad \textit{so by interpolation} \quad u \in C_t^{\frac{\beta}{1+\beta}} C_x^1.$$

In particular, if $\alpha \geq \frac{1}{2}$ then $\nabla u \in C_t^{\frac{2}{3}-} C_x^0$.

Proof. Since $\nabla u(t)$ are in $C^\beta(\overline{\Omega(u(t))})$ uniformly in t we can use the uniform convergence of the difference quotients and the Lipschitz continuity in time of $u(t)$ from Lemma 4.23

$$\begin{aligned}(\nabla u(t,x) - \nabla u(s,x)) \cdot e &= \frac{1}{h}(u(t,x+he) - u(t,x)) + \frac{1}{h}(u(s,x+he) - u(s,x)) + O(h^\beta)\\ &= O(h^{-1}|t-s|) + O(h^\beta).\end{aligned}$$

choose $h = |t-s|^{\frac{1}{1+\beta}}$ to match the orders of the two terms. □

4.5. **Comparison principle in star-shaped setting.** Now we state a comparison principle that closes our discussion on equivalences of notions. We will utilize this comparison principle later on in Section 6, when we discuss a specific energy solution generated by minimizing movements.

Proposition 4.26. *Suppose that w and u are respectively a subsolution and a LSC supersolution (or a supersolution and an USC subsolution) of* (M) *in $[0,T]$ with strongly star-shaped initial data, with the ordering $w(0) \le u(0)$ (or $w(0) \ge u(0)$). Further suppose that w satisfies the following:*

- *(a) $\{w(t) > 0\}$ is bounded and strongly star-shaped at each time, and its boundary is uniformly $C^{1,\alpha}$ in space and Lipschitz in time.*
- *(b) There exists a finite partition $\{t_i\}_{i=0}^{i=n}$ of $[0,T]$ such that w is monotone in $[t_i, t_{i+1}]$.*

Then $w \le u$ (or $w \ge u$).

As is typical with viscosity solutions, one of the main difficulties in the comparison principle of weak solutions is the lack of regularity. We follow a typical approach based on sup / inf convolutions in order to regularize at the point where the first touching occurs. The novel difficulty has to do with the weak notion of positive velocity and the difficulties stemming from crossings which occur with zero velocity. We can use sup / inf convolutions with decreasing radius to accelerate the free boundary velocity, but a delicate argument relying on the regularity of the (O) solution is still needed to actually locate a positive velocity first touching point.

Let us mention that an (OVS) solution with star-shaped initial data satisfies (a) in Proposition 4.26 due to Lemma 4.22 and Lemma 4.23 and (b) due to Remark 4.2. It is also an (M) by Lemma 4.20. Thus, from the comparison principle Proposition 4.26, we can derive the uniqueness:

Corollary 4.27. *Any motion law solution* (M) *with a strongly star-shaped initial data is the unique star-shaped obstacle solution* (O) *with the same initial data.*

Proof of Proposition 4.26. Let us assume that w is a (M) subsolution with regularity properties, and u is a LSC (M) supersolution. We will show that

$$w(t, \lambda\cdot) \le u(t,\cdot) \text{ for } t > 0 \text{ and } \lambda > 1,$$

which yields $w \le u$ in the limit $\lambda \searrow 1$. The proof for the other half of the statement is parallel, arguing the opposite inequality with $\lambda < 1$.

Fix $\lambda > 1$. Define

$$W(t,x) := \sup_{y \in \overline{B}_{\rho(t)}(x)} w(t, \lambda y).$$

Here $\rho(t) = \rho_0 e^{-t}$, and ρ_0 is chosen proportional to $(\lambda - 1)$ so that $W(0) < u(0)$ in $\overline{\{W(0) > 0\}}$, which is possible by strong star-shapedness. The main motivation for the sup-convolution is the effective reduction of the normal velocity of W by $r'(t) < 0$ compared to w. Hence if W and u touch each other, either the normal velocity of u is negative or w has positive normal velocity near the contact, and therefore the dynamic slope condition can be used in either situation to arrive at a contradiction. Recall that W is subharmonic in its positive set.

We would like to show that $W(t) < u(t)$ in $\overline{\{W(t) > 0\}}$ for all $t \geq 0$. Define the first contact time

$$t_0 = \inf \left\{ t > 0 : W(t) \not< u(t) \text{ in } \overline{\{W(t) > 0\}} \right\}.$$

Clearly $t_0 > 0$ by lower semi-continuity of u in time. If t_0 is finite, we want to get a contradiction. The only non-standard part here lies in the fact that the support of u may jump in time, so let us discuss this point carefully. Since we assume that W is star-shaped, W and its support continuously decrease as λ increases. The support of $\{W(t_0) > 0\}$ indeed can be made arbitrarily close to $\overline{B}_{\rho(t_0)}(0)$, in particular a subset of the fixed boundary $U^{\complement}$ for u, if λ is sufficiently large. Thus if $\{W(\cdot, t_0) > 0\} \not\subset \{u(\cdot, t_0) > 0\}$ due to discontinuity in time in u at t_0, we increase λ in the definition of W until $\{W(t_0) > 0\}$ precisely touches $\{u(t_0) > 0\}$ from inside, so that there exists a point $x_0 \in \partial\{W(t_0) > 0\} \cap \partial\{u(t_0) > 0\}$. We point out that ρ_0 does not need to be changed even if we need to increase λ. Note that $W \leq u$ for $t \leq t_0$ for this larger choice of λ due to the definition of t_0 and $W(t_0) - u(t_0)$ being subharmonic in $\{W(t_0) > 0\}$. In particular $u(t_0) - W(t_0)$ has a minimum at x_0.

By the definition of W there is a point $y_0 \in \partial\{w(t_0, \lambda\cdot) > 0\}$ with $y_0 \in \overline{B_{\rho(t_0)}}(x_0)$, and

$$W(t_0, x) \geq w(t_0, \lambda(x - x_0 + y_0)) =: \phi(x).$$

As $u(t_0) - \phi$ has a minimum at x_0, we deduce that

$$\lambda \nabla w(t_0, \lambda y_0) = \nabla \phi(x_0) \in D_- u(x_0). \tag{4.7}$$

By our assumption (b), we know that w is either monotone increasing or decreasing in a closed time interval $[t_{-1}, t_0]$ with $t_{-1} < t_0$. If w is monotone decreasing in $[t_{-1}, t_0]$, then W has strictly decreasing support in $[t_{-1}, t_0]$, namely for $c = -r'(t_0) > 0$ we have

$$B_{c(t_0 - t)}(x_0) \subset \{W(t) > 0\} \subset \{u(t) > 0\} \text{ for } t_{-1} \leq t < t_0. \tag{4.8}$$

In particular the negative cone condition (4.3) is satisfied for u at (t_0, x_0), and so as u is a (M) supersolution we conclude by (4.7) that $\lambda^2 |\nabla w|^2(t_0, \lambda y_0) \leq 1 - \mu_-$. However as w is itself (M) subsolution, the stability condition yields $\lambda^2 |\nabla w|^2(t_0, \lambda y_0) \geq \lambda^2(1 - \mu_-)$, a contradiction.

Now suppose w is monotone increasing in $[t_{-1}, t_0]$. It is possible that in this case u still satisfies the negative cone condition (4.3) for some $c < 0$ and $r_0 > 0$, in which case one can still proceed as in the above case. So now suppose that (4.3) is not true for $c := -\frac{1}{3}\rho'(t_0) > 0$ and any $r_0 > 0$. This means that along an increasing time sequence t_n converging to t_0, there is x_n such that

$$x_n \in \{u(t_n) = 0\} \cap B_{c(t_0 - t_n)}(x_0).$$

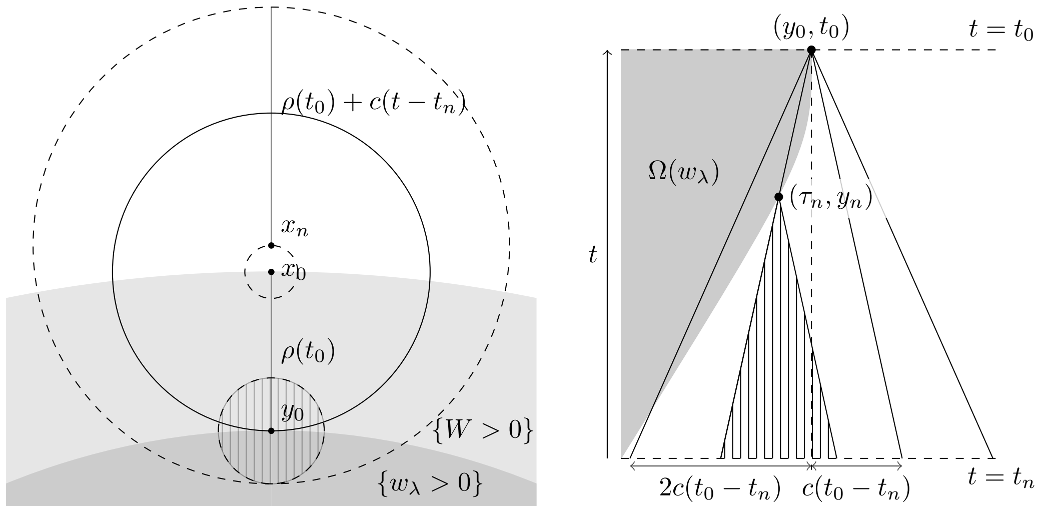

FIGURE 7. Left: spatial picture of the sup convolution balls at time t_0 and t_n overlayed, patterned ball is in the zero level set of w_λ at time t_n. Right: space-time picture displaying the definition of (τ_n, y_n).

Now we consider what this means in terms of w. Recall y_0 above and consider y such that $|y - y_0| \leq 2c(t_0 - t_n)$. Then

$$\begin{aligned}|y - x_n| &\leq |y - y_0| + |y_0 - x_0| + |x_0 - x_n| \\ &\leq 2c(t_0 - t_n) + \rho(t_0) + c(t_0 - t_n) \\ &\leq \rho(t_n),\end{aligned}$$

where the last inequality is by convexity of $\rho(t)$. Thus, by definition of W, $w(t_n, \lambda y) \leq W(t_n, x_n) \leq u(t_n, x_n) = 0$. Namely $\overline{B}_{2c(t_0-t_n)}(y_0) \subset \{w(t_n, \lambda\cdot) = 0\}$. See the left image in Figure 7.

Now let

$$\tau_n := \sup\{s \in [t_n, t_0) : \overline{B}_{c(t_0-s)}(y_0) \subset \{w(s, \lambda\cdot) = 0\}\}.$$

See the right image in Figure 7. Since $\{w(t) = 0\}$ evolves continuously in time (Corollary 4.24), we have $\tau_n > t_n$, and if $\tau_n < t_0$ then there is a contact point $y_n \in \partial\{w(\tau_n) > 0\} \cap \partial B_{c(t_0-\tau_n)}(y_0)$. If the order is preserved up to $t = t_0$, i.e. $\tau_n = t_0$, then we set $y_n = y_0$. From the definition of (y_n, τ_n) we see that positive cone condition (4.2) holds for $w(\cdot, \lambda\cdot)$ at (τ_n, y_n) and therefore for w at $(t_n, \lambda y_n)$ with λc. By (M) we have $|\nabla w|^2(t_n, \lambda y_n) \geq 1 + \mu_+$.

Since $(\tau_n, y_n) \to (t_0, y_0)$, by the gradient continuity in Corollary 4.25, we deduce

$$\lambda^2 |\nabla w|^2(t_0, \lambda y_0) = \lim_{n\to\infty} \lambda^2 |\nabla w|^2(\tau_n, \lambda y_n) \geq \lambda^2(1 + \mu^+).$$

But this contradicts the fact that u is a (M) supersolution and hence by (4.7) we must have $\lambda^2 |\nabla w|^2(\lambda y_0, t_0) \leq 1 + \mu_+$. Hence we conclude. □

5. Basic space and time regularity properties of energy solutions

In this section we study energy solutions (E) establishing existence and various spatial and temporal regularity properties. We will work in a general setting, without the special geometry of star-shapedness which was used in the previous section.

In Section 5.1 we recall several results from the literature on inwards and outwards minimizers of one-phase functionals. These results, in particular, also apply to states satisfying the global stability condition of energy solutions (E).

In Section 5.2 we show BV time regularity of energy solutions using a typical Grönwall argument via the energy dissipation inequality, see Lemma 5.7. With this regularity we can also establish that the energy dissipation balance holds with equality, see Lemma 5.10.

Finally in Section 5.3 we establish the main temporal regularity result of the section, Proposition 5.16. Abstract theory of bounded variation maps from $[0,T]$ into a metric space shows that $u(t)$ has left and right limits $u_\ell(t)$ and $u_r(t)$ at every time. Then we use the monotonicity structure of the problem to show that the lower and upper-semicontinuous envelopes of an energy solution are exactly $u_\ell(t) \wedge u_r(t)$ and $u_\ell(t) \vee u_r(t)$ and these are energy solutions as well. This is independently interesting, elucidating the structure of the jump discontinuities, and it also plays a key role in establishing the dynamic slope condition in Section 7.

5.1. Inwards and outwards minimality and spatial regularity.

Let V be an open region that contains $\mathbb{R}^d \setminus U$. In this section we will discuss the regularity of minimizers of the *dissipation augmented energy*, defined above in (1.3),

$$\mathcal{E}(V,u) := \mathcal{J}(u) + \mathrm{Diss}(V, \Omega(u)).$$

We will also abuse notation to write $\mathcal{E}(v,u) = \mathcal{E}(\Omega(v),u)$ when v is another nonnegative function on $\mathbb{R}^n$.

Note that if u is an energy solution on $[0,T]$, from the global stability property at each time $u(t)$ is a minimizer for $\mathcal{E}(u(t),\cdot)$.

It is convenient to make a connection with the notions of inwards and outwards minimality for the Bernoulli functional $\mathcal{J}$. First of all we introduce the notation for $Q > 0$ and an open region $U \subset \mathbb{R}^d$

$$\mathcal{J}_Q(u) := \int_U |\nabla u|^2 + Q\mathbf{1}_{\{u>0\}}\, dx.$$

We have written and will continue to write $\mathcal{J} = \mathcal{J}_1$ abusing notation.

Next we introduce the notions of inwards and outwards minimizers. These notions have appeared in the literature before, for example see the book [36] for further references.

Definition 5.1. $u \in H^1(U)$ is an *outward (resp. inward) minimizer* of $\mathcal{J}_Q(\cdot;U)$ if

(1) The set $\Omega(u) = \{u > 0\}$ is open and u is harmonic in $\Omega(u)$.
(2) For any $v \in u + H^1_0(U)$ with $v \geq u$ (resp. $v \leq u$)

$$\mathcal{J}_Q(u) \leq \mathcal{J}_Q(v).$$

Minimizers of $\mathcal{E}(V,\cdot)$ have a natural inward/outward $\mathcal{J}_{1\pm\mu_\pm}$ minimality property.

Lemma 5.2. *Let V be an open set that contains $\mathbb{R}^d \setminus U$. Suppose that u is a global minimizer of*

$$\mathcal{J}(v) + \mathrm{Diss}(V, \Omega(v)) \quad over \quad v \in u + H^1_0(U).$$

Then u is an outwards minimizer for $\mathcal{J}_{1+\mu_+}$ and an inwards minimizer for $\mathcal{J}_{1-\mu_-}$. Furthermore if B_r lies outside of $\overline{V}$ then u minimizes $\mathcal{J}_{1+\mu_+}$ in B_r, and if $B_r \subset V$ then u minimizes $\mathcal{J}_{1-\mu_-}$ in B_r.

Proof. Suppose $v \in u + H^1_0(U)$ with $v \geq u$. Then $\{v > 0\} \supset \{u > 0\}$, so

$$|V \setminus \Omega(u)| \geq |V \setminus \Omega(v)| \quad \text{and} \quad |\Omega(u) \cap V| \leq |\Omega(v) \cap V|. \tag{5.1}$$

We can apply (5.1) along with the minimality property of u to obtain

$$\begin{aligned}
\mathcal{J}_{1+\mu_+}(u) &= \mathcal{J}_1(u) + \mu_+|\Omega(u)| \\
&= \mathcal{J}_1(u) + \mu_+|\Omega(u) \setminus V| + \mu_+|\Omega(u) \cap V| \\
&= \mathcal{J}_1(u) + \mathrm{Diss}(V, \Omega(u)) - \mu_-|V \setminus \Omega(u)| + \mu_+|\Omega(u) \cap V| \\
&\leq \mathcal{J}_1(v) + \mathrm{Diss}(V, \Omega(v)) - \mu_-|V \setminus \Omega(v)| + \mu_+|\Omega(v) \cap V| \\
&= \mathcal{J}_{1+\mu_+}(v).
\end{aligned}$$

Lastly note that if $v = u$ outside of $B_r \subset \mathbb{R}^d \setminus V$ then (5.1) holds with equalities.

Symmetrical computations give the (inwards) minimality property for $\mathcal{J}_{1-\mu_-}$. □

5.1.1. *Viscosity solution properties of inward / outward minimizers.* The notions of inwards and outwards minimizers are reminiscent of viscosity sub and supersolutions. Indeed there is a direct correspondence between the two notions.

Lemma 5.3. *Suppose that $S > 0$ and u is an inwards minimizer of $\mathcal{J}_S$ in a domain U. Then in the viscosity sense*

$$|\nabla u|^2 \geq Q \quad on \ \partial\{u > 0\} \cap U.$$

Similarly, if u is an outwards minimizer of $\mathcal{J}_S$ then in the viscosity sense

$$|\nabla u|^2 \leq Q \quad on \ \partial\{u > 0\} \cap U.$$

In general the implication cannot go the other way, there are viscosity solutions, i.e. stationary points of the energy, which are not energy minimizers.

It is well known that such inwards and outwards minimality properties imply the corresponding viscosity solution conditions, see for example [36, Proposition 7.1]. We present a proof anyway since we will use similar, but more involved, computations in the proof of Theorem 7.1.

Before proceeding with the proof let us write down a corollary of Lemma 5.2 and Lemma 5.3 for energy solutions.

Corollary 5.4 (Stability implies slope in pinning interval)**.** *Suppose that u is an energy solution on $[0, T]$, then for each time $u(t)$ is a viscosity solution of*

$$1 - \mu_- \leq |\nabla u(t)|^2 \leq 1 + \mu_+ \quad on \ \partial\{u(t) > 0\} \cap U.$$

Thus energy solutions satisfy the stability condition of (M).

Proof of Lemma 5.3. **(Subsolution)** Suppose that a smooth test function φ touches u from above in $\overline{\Omega(u)}$ strictly at $x_0 \in \partial\Omega(u)$ with $p := \nabla\varphi(x_0) \neq 0$ and $\Delta\varphi(x_0) < 0$. Let $\delta > 0$ and consider the comparison function

$$u_\delta(x) := u(x) \wedge (\varphi(x) - \delta)_+.$$

Since the touching is strict $\{u > 0\} \setminus \{u_\delta > 0\}$ is contained in a ball of radius $o_\delta(1)$ around x_0 as $\delta \to 0$. In particular we can assume that δ is small enough so that φ is superharmonic in $\{u_\delta < u\}$. By the inwards minimality of u

$$\mathcal{J}_Q(u) \leq \mathcal{J}_Q(u_\delta).$$

Now note that u_δ extends to a superharmonic function in $\{u > 0\}$ by

$$\bar{u}_\delta(x) := \begin{cases} u_\delta(x) & x \in \Omega(u_\delta) \\ \varphi(x) - \delta & x \in \Omega(u) \setminus \Omega(u_\delta). \end{cases}$$

So we can apply Lemma A.2, formula (A.2), and we find

$$\begin{aligned} \mathcal{J}_Q(u) - \mathcal{J}_Q(u_\delta) &\geq \int_{\{u>0\}\setminus\{u_\delta>0\}} Q - |\nabla\varphi(x)|^2 \, dx \\ &\geq [(Q - |p|^2) - o_\delta(1)]|\{u > 0\} \setminus \{u_\delta > 0\}| \end{aligned}$$

Combining the previous

$$[(Q - |p|^2) - o_\delta(1)]|\{u > 0\} \setminus \{u_\delta > 0\}| \leq 0$$

so dividing through by $|\{u > 0\} \setminus \{u_\delta > 0\}| > 0$ and taking $\delta \to 0$ we find

$$Q \leq |p|^2.$$

(Supersolution) Suppose that a smooth test function φ touches u from below strictly at $x_0 \in \partial\Omega(u)$ with $p := \nabla\varphi(x_0) \neq 0$ and $\Delta\varphi(x_0) > 0$. Let $\delta > 0$ and consider the comparison function

$$u_\delta(x) = u(x) \vee (\varphi(x) + \delta).$$

Since the touching is strict $\{u_\delta > u\}$ is contained in a ball of radius $o_\delta(1)$ around x_0 as $\delta \to 0$. In particular we can assume that δ is small enough so that $\Delta u_\delta \geq 0$. By the outwards minimality property of u

$$\mathcal{J}_S(u) \leq \mathcal{J}_S(u_\delta).$$

Applying Lemma A.2 (note u_δ not harmonic in $\{u_\delta > 0\}$ but in general subharmonic) we find

$$\begin{aligned} \mathcal{J}_S(u) - \mathcal{J}_S(u_\delta) &\geq \int_{\{u_\delta>0\}\setminus\{u>0\}} |\nabla u_\delta|^2 - Q \, dx \\ &= \int_{\{u_\delta>0\}\setminus\{u>0\}} |\nabla\varphi(x)|^2 - Q \, dx \\ &\geq [(|p|^2 - Q) - o_\delta(1)]|\{u_\delta > 0\} \setminus \{u > 0\}| \end{aligned}$$

Combining the previous

$$[(|p|^2 - Q) - o_\delta(1)]|\{u_\delta > 0\} \setminus \{u > 0\}| \leq 0$$

so dividing through by $|\{u_\delta > 0\} \setminus \{u > 0\}| > 0$ and taking $\delta \to 0$ we find

$$|p|^2 \leq Q.$$

□

5.1.2. *Regularity properties of inward / outward minimizers.* Next we collect several basic regularity results of $\mathcal{J}_Q$ inwards / outwards minimizers.

Lemma 5.5. *(i) (Lipschitz estimate) There is $C(d) \geq 1$ so that for any viscosity solution u of*

$$\Delta u = 0 \quad in \quad \Omega(u) \cap B_2 \quad and \quad |\nabla u|^2 \leq Q \quad on \quad \partial\Omega(u) \cap B_2$$

we have

$$\|\nabla u\|_{L^\infty(B_1)} \leq C(\sqrt{Q} + \|\nabla u\|_{L^2(B_2)}).$$

In particular this holds for outward minimizers of $\mathcal{J}_Q$ in B_2 by Lemma 5.3.

(ii) (Non-degeneracy) There is $c(d, Q) > 0$ so that if $u \in H^1(U)$ is an inward minimizer of $\mathcal{J}_Q$ in U, then

$$\sup_{B_r(x)} u \geq cr \quad \text{for any } x \in \partial\{u > 0\} \text{ and } B_r(x) \subset U.$$

(iii) (Density estimates) For $Q_1 \leq Q_2$, there is $c(d, Q_1, Q_2) > 0$ so that if u is an inward minimizer of $\mathcal{J}_{Q_1}$ and an outward minimizer of $\mathcal{J}_{Q_2}$ in U then

$$c \leq \frac{|\Omega(u) \cap B_r(x)|}{|B_r|} \leq 1 - c \text{ for any } x \in \partial\Omega(u) \text{ with } B_r(x) \subset U.$$

(iv) (Perimeter estimate) There is $C(d) > 0$ such that if u is an inward minimizer of $\mathcal{J}_S$ in B_2 then

$$\operatorname{Per}(\Omega(u); B_1) \leq C\sqrt{Q}(1 + \|\nabla u\|_{L^2(B_2)}).$$

(v) (Hausdorff dimension) If u satisfies the conclusions of (iii) and perimeter estimate (iv) in B_2 then

$$\mathcal{H}^{d-1}(\partial\Omega(u) \cap B_1) \leq C(1 + \|\nabla u\|_{L^2(B_2)})$$

where C depends on the constants in (iii) and (iv).

Proof. See the books [6, 36] for presentations of the proofs and citations for their original appearances in the literature, Lipschitz continuity of viscosity solutions is in [6, Lemma 11.19], non-degeneracy is in [36, Lemma 4.4], density estimates are in [36, Lemma 5.1], perimeter estimates are in [36, Lemma 5.6], Hausdorff dimension estimates in [36, Lemma 5.9]. □

Remark 5.6. The $C^{1,\beta}$ regularity of $\mathcal{E}(u, \cdot)$ minimizers is known due to the very recent result of Ferreri and Velichkov [22] when $\{u > 0\}$ is $C^{1,\alpha}$.

5.2. **Bounded variation regularity in t.** BV regularity in time is typical for solutions of (global) dissipative evolution problems [30]. Following the standard argument, we establish the BV in time regularity using a Grönwall type argument with the energy dissipation inequality.

Lemma 5.7. *Suppose u is an energy solution (E) on $[0, T]$. Then*

$$\mathbf{1}_{\Omega(u(t))} \in \mathrm{BV}([0, T]; L^1(\mathbb{R}^d))$$

and

$$\mathcal{J}(u(t)), \ P(u(t)) \in \mathrm{BV}([0, T]; \mathbb{R}),$$

where recall the pressure is $P(u(t)) = \int_{\partial\Omega(t)} |\nabla u| dS = \int_{\partial U} \frac{\partial u}{\partial n} dS = F(t)^{-1} \int |\nabla u(t)|^2 \, dx$.

Before beginning the proofs we make a remark about the definition of energy solution.

Remark 5.8. Energy solutions will also satisfy an *energy dissipation balance* condition. Define the total variation of the dissipation distance along a path $\Lambda(t)$ of finite measure subsets of $\mathbb{R}^d$

$$\overline{\mathrm{Diss}}(\Lambda(\cdot);[s,t]) := \sup\left\{\sum_{k=1}^{N} \mathrm{Diss}(\Lambda(t_{k-1}),\Lambda(t_k)) : (t_k)_{k=0}^{N} \text{ partitions } [s,t]\right\}. \tag{5.2}$$

Then let $u(t)$ be an energy solution on $[0,T]$, i.e. satisfying the forcing, stability, and energy dissipation inequality conditions of (E). One can show immediately, by applying (1.6) on each subinterval of an arbitrary partition of $[t_0,t_1]$, that

$$\mathcal{J}(u(t_0)) - \mathcal{J}(u(t_1)) + \int_{t_0}^{t_1} 2\dot{F}(t)P(t)\ dt \geq \overline{\mathrm{Diss}}(u(\cdot);[t_0,t_1]). \tag{5.3}$$

With more work (see Lemma 5.10 later) combining with the stability property one can also show the identity

$$\mathcal{J}(u(t_0)) - \mathcal{J}(u(t_1)) + \int_{t_0}^{t_1} 2\dot{F}(t)P(t)\ dt = \overline{\mathrm{Diss}}(u(\cdot);[t_0,t_1]). \tag{5.4}$$

In some works, for example [1], the energy dissipation balance (5.4) is used in place of the energy dissipation inequality (1.6) as part of the definition of energy solution. This difference of definition is just a matter of preference at least in this problem.

Remark 5.9. It is quite natural, and important for applications, to consider general $F(t,x)$ and allow for F to be only BV regular in time. However, this generality does add serious complications which would significantly lengthen the presentation, and it is not so relevant to the goals of the present work.

Proof of Lemma 5.7. By the energy dissipation inequality (5.3)

$$\mathcal{J}(u(0)) - \mathcal{J}(u(t)) + \int_0^t 2\dot{F}(s)P(u(s))\ ds \geq \overline{\mathrm{Diss}}(\Omega(u(\cdot));[0,t]) \geq \mu_+ \wedge \mu_- [\mathbf{1}_{\Omega(\cdot)}]_{\mathrm{BV}([0,t];L^1)} \tag{5.5}$$

In the remainder of the proof we will denote the Dirichlet energy

$$\mathcal{D}(t) := \int_{\Omega(u(t))} |\nabla u(t,x)|^2\ dx = F(t)P(u(t)).$$

Note that $\mathcal{D}(t) \leq \mathcal{J}(u(t))$ so in particular

$$\begin{aligned}\mathcal{D}(t) &\leq \mathcal{J}(u(0)) + \int_0^t 2\dot{F}(s)P(u(s))\ ds \\ &= \mathcal{J}(u(0)) + \int_0^t 2\frac{d}{ds}(\log F)(s)\mathcal{D}(s)\ ds\end{aligned}$$

for all $0 \leq t \leq T$ so by Grönwall

$$\mathcal{D}(t) \leq \mathcal{J}(u(0))\frac{F(t)^2}{F(0)^2}.$$

Thus $P(u(t)) \leq F(0)^{-2}F(t)$, and from (5.5) we have

$$\begin{aligned}\mathcal{J}(u(t)) + \mu_+ \wedge \mu_-[\mathbf{1}_{\Omega(\cdot)}]_{\mathrm{BV}([0,t];L^1)} &\leq \mathcal{J}(u(0))\left(1 + \frac{1}{F(0)^2}\int_0^t 2\dot{F}(s)F(s)\ ds\right) \\ &= \mathcal{J}(u(0))\left(1 + \frac{F(t)^2 - F(0)^2}{F(0)^2}\right) \\ &= \mathcal{J}(u(0))\frac{F(t)^2}{F(0)^2}.\end{aligned}$$

To summarize,

$$\mathcal{J}(u(t)) + \mu_+ \wedge \mu_-[\mathbf{1}_{\Omega(\cdot)}]_{\mathrm{BV}([0,t];L^1)} \leq \mathcal{J}(u(0))\frac{F(t)^2}{F(0)^2}. \tag{5.6}$$

In particular

$$[\mathbf{1}_{\Omega(\cdot)}]_{\mathrm{BV}([0,t];L^1)} \leq \frac{1}{\mu_+ \wedge \mu_-}\mathcal{J}(u(0))\frac{F(t)^2}{F(0)^2}. \tag{5.7}$$

This gives the BV in time estimate of $\mathbf{1}_{\Omega(t)}$.

Now we turn to the BV estimate of the Dirichlet energy. Focusing on the other term on the left hand side in (5.6), and using the fact that $\mathcal{J}(u(t)) = \mathcal{D}(t) + |\Omega(u(t))|$,

$$\mathcal{D}(t) \leq \frac{F(t)^2}{F(0)^2}\mathcal{D}(0) + \frac{F(t)^2}{F(0)^2}|\Omega(u(0))| - |\Omega(u(t))|$$

or rearranging this

$$\begin{aligned}\mathcal{D}(t) - \mathcal{D}(0) &\leq \frac{\mathcal{D}(0)}{F(0)^2}(F(t)^2 - F(0)^2) + \frac{1}{F(0)^2}(F(t)^2 - F(0)^2)|\Omega(u(0))| \\ &\quad + |\Omega(u(t))\Delta\Omega(u(0))| \\ &= \frac{\mathcal{J}(u(0))}{F(0)^2}(F(t)^2 - F(0)^2) + |\Omega(u(t))\Delta\Omega(u(0))|.\end{aligned} \tag{5.8}$$

Global stability implies a bound in the other direction

$$\mathcal{J}(u(0)) \leq J\left(\frac{F(0)}{F(t)}u(t)\right) + \mu_+ \vee \mu_-|\Omega(u(0))\Delta\Omega(u(t))|,$$

which expands out as

$$\mathcal{D}(0) + |\Omega(u(0))| \leq \frac{F(0)^2}{F(t)^2}\mathcal{D}(t) + |\Omega(u(t))| + \mu_+ \vee \mu_-|\Omega(u(0))\Delta\Omega(u(t))|,$$

which we can use to find

$$\mathcal{D}(0) - \mathcal{D}(t) \leq \frac{\mathcal{D}(t)}{F(t)^2}(F(0)^2 - F(t)^2) + (1 + \mu_+ \vee \mu_-)|\Omega(u(0))\Delta\Omega(u(t))|. \tag{5.9}$$

Since 0 and t replaced by arbitrary $t_0 < t_1 \in [0,T]$, for any partition of $[0,T]$ we can apply (5.9) and (5.8) together to find:

$$\begin{aligned}\sum_j |\mathcal{D}(t_{j+1}) - \mathcal{D}(t_j)| \leq \sum_j \Bigg[&\max\left\{\frac{\mathcal{J}(u(t_j))}{F(t_j)^2}, \frac{\mathcal{J}(u(t_{j+1}))}{F(t_{j+1})^2}\right\}\left|F(t_{j+1})^2 - F(t_j)^2\right| \\ &+ (1 + \mu_+ \vee \mu_-)|\Omega(u(t_{j+1}))\Delta\Omega(u(t_j))|.\end{aligned}$$

Applying (5.6) again for $F(t)^{-2}\mathcal{J}(u(t)) \le F(0)^{-2}\mathcal{J}(u(0))$, we obtain that

$$\sum_j |\mathcal{D}(t_{j+1}) - \mathcal{D}(t_j)| \le \frac{\mathcal{J}(u(0))}{F(0)^2}[F^2]_{\mathrm{BV}([0,T])} + (1+\mu_+ \vee \mu_-)[\mathbf{1}_{\Omega(\cdot)}]_{\mathrm{BV}([0,T];L^1)}.$$

Thus we conclude that $\mathcal{D}(t) \in BV([0,T])$ with the estimate, applying (5.7),

$$[\mathcal{D}]_{BV([0,T])} \le \left[[F^2]_{\mathrm{BV}([0,T])} + \frac{1+\mu_+\vee\mu_-}{\mu_+\wedge\mu_-}F(t)^2\right]\frac{\mathcal{J}(u(0))}{F(0)^2}.$$

Finally since $\mathcal{J}(u(t)) = \mathcal{D}(t) + |\Omega(u(t))|$ and both terms on the right are in $BV([0,T])$ then so is $\mathcal{J}(u(t))$. Similarly $P(u(t)) = F(t)^{-1}\mathcal{D}(t)$ is in $BV([0,T])$ since $F(t)$ is bounded from below. □

From these regularity properties we can establish that the stability and energy dissipation inequality properties of energy solutions force the *energy dissipation equality*.

Lemma 5.10 (Energy dissipation equality)**.** *If u is an energy solution on $[0,T]$ then for all $s \le t$ in $[0,T]$*

$$\mathcal{J}(u(s)) - \mathcal{J}(u(t)) + \int_s^t 2\dot{F}(\tau)P(u(\tau))\, d\tau = \overline{\mathrm{Diss}}(\Omega(u(\cdot)); [s,t]).$$

Proof. We need to establish the upper bound to complement (1.6). This will follow from the time regularity and the global stability property. We may take $s = 0$ and $t = T$. Let $\varepsilon > 0$. By the continuity of $\log F(t)$, we can choose a (finite) partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$ such that

$$\sup_j (t_{j+1} - t_j) \le \varepsilon \quad \text{and} \quad \sup_{t\in[t_j,t_{j+1}]} \frac{F(t)}{F(t_{j+1})} \le 1+\varepsilon. \tag{5.10}$$

We write the energy dissipation using a telescoping sum

$$\mathcal{J}(u(0)) - \mathcal{J}(u(T)) = \sum_{j=0}^{N-1}[\mathcal{J}(u(t_j)) - \mathcal{J}(u(t_{j+1}))]. \tag{5.11}$$

Applying the global stability condition for $u(t_j)$,

$$\begin{aligned}\mathcal{J}(u(t_j)) - \mathcal{J}(u(t_{j+1})) &\le \mathcal{J}\left(\tfrac{F(t_j)}{F(t_{j+1})}u(t_{j+1})\right) + \mathrm{Diss}(\Omega(t_j), \Omega(t_{j+1})) - \mathcal{J}(u(t_{j+1})) \\ &= \mathrm{Diss}(\Omega(t_j),\Omega(t_{j+1})) + \underbrace{(\tfrac{F(t_j)^2}{F(t_{j+1})^2} - 1)\mathcal{D}(u(t_{j+1}))}_{:=\mathrm{A}}\end{aligned}$$

Now from the fact that $\mathcal{D}(u(t)) = F(t)P(u(t))$,

$$A = \frac{F(t_j)^2 - F(t_{j+1})^2}{F(t_{j+1})}P(u(t_{j+1})) = \int_{t_j}^{t_{j+1}} 2\frac{F(t)}{F(t_{j+1})}\dot{F}(t)P(u(t_{j+1}))\, dt.$$

From (5.10) we have

$$
\begin{aligned}
A &= \int_{t_j}^{t_{j+1}} 2\frac{F(t)}{F(t_{j+1})}\dot{F}(t)P(u(t_{j+1}))\ dt \\
&\leq \int_{t_j}^{t_{j+1}} 2\frac{F(t)}{F(t_{j+1})}\dot{F}(t)P(u(t))\ dt + 2(1+\varepsilon)\|\dot{F}\|_\infty[P(u(\cdot))]_{\mathrm{BV}((t_j,t_{j+1}])}(t_{j+1}-t_j) \\
&\leq (1+\varepsilon)\int_{t_j}^{t_{j+1}} 2\dot{F}(t)P(u(t))\ dt + 2(1+\varepsilon)\varepsilon\|\dot{F}\|_\infty[P(u(\cdot))]_{\mathrm{BV}((t_j,t_{j+1}])}
\end{aligned}
\tag{5.12}
$$

Combining the previous estimates into (5.11) we find

$$
\begin{aligned}
\mathcal{J}(u(0)) - \mathcal{J}(u(T)) &\leq (1+\varepsilon)\int_0^T 2\dot{F}(t)P(u(t))\ dt + \sum_{j=0}^{N-1}\mathrm{Diss}(\Omega(t_j),\Omega(t_{j+1})) \\
&\qquad + \sum_{j=0}^{N-1} 2(1+\varepsilon)\varepsilon\|\dot{F}\|_\infty[P(u(\cdot))]_{\mathrm{BV}((t_j,t_{j+1}])} \\
&\leq \overline{\mathrm{Diss}}(\Omega(u(\cdot));(0,T)) + (1+\varepsilon)\int_0^T 2\dot{F}(t)P(u(t))\ dt \\
&\qquad + 2(1+\varepsilon)\varepsilon\|\dot{F}\|_\infty[P(u(\cdot))]_{\mathrm{BV}((0,T])}.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary we get the result.

□

5.3. **Left and right limits and monotonicity of jumps.** The BV time regularity of $\mathbf{1}_{\Omega(u(t))}$ allows us to establish further temporal left and right continuity properties of the evolution at every time. Recall the dissipation augmented energy functional (1.3),

$$\mathcal{E}(V,u) := \mathcal{J}(u) + \mathrm{Diss}(V,\Omega(u)).$$

First we discuss a monotonicity property for minimizers of $\mathcal{E}$.

5.3.1. *No crossing property of $\mathcal{E}$ minimizers.* The next lemma shows that minimizers of the dissipation distance augmented energy $\mathcal{E}$, defined in (1.3), satisfy a certain ordering property.

Suppose that u_1 and u_2 are arbitrary $H^1(U)$ functions. Note that

$$\mathcal{J}(u_1\vee u_2) + \mathcal{J}(u_1\wedge u_2) = \mathcal{J}(u_1) + \mathcal{J}(u_2)$$

and

$$\mathrm{Diss}(V,\Omega(u_1\vee u_2)) + \mathrm{Diss}(V,\Omega(u_1\wedge u_2)) = \mathrm{Diss}(V,\Omega(u_1)) + \mathrm{Diss}(V,\Omega(u_2)).$$

Combining the previous two equations yields

$$\mathcal{E}(V,u_1\vee u_2) + \mathcal{E}(V,u_1\wedge u_2) = \mathcal{E}(V,u_1) + \mathcal{E}(V,u_2). \tag{5.13}$$

It is standard to derive from this additivity property that the pointwise minimum and maximum of two minimizers with the same Dirichlet condition are energy minimizers as well. Adding in the strong maximum principle for harmonic functions we can show an ordering property of the collection of energy minimizers, again this is a sufficiently standard idea that we do not have a particular original reference.

Lemma 5.11 (No crossing)**.** *Let V be an open set containing $\mathbb{R}^d \setminus U$. If u_1 and u_2 both minimize $\mathcal{E}(V, \cdot)$ with respect to $H^1_0(U)$ perturbations and $u_1 - u_2 \in H^1_0(U)$. Then $u_1 \wedge u_2$ and $u_1 \vee u_2$ are also minimizers and u_1 and u_2 are ordered in each connected component of $\Omega(u_1) \cup \Omega(u_2) = \Omega(u_1 \vee u_2)$.*

Remark 5.12. To be clear the ordering between u_1 and u_2 may differ in the different connected components of $\Omega(u_1) \cup \Omega(u_2)$. For example, given two ordered minimizers the same scenario can be repeated far away with the reverse ordering to create unordered minimizers.

Proof of Lemma 5.11. Recalling (5.13) we have

$$\mathcal{E}(V, u_1 \vee u_2) + \mathcal{E}(V, u_1 \wedge u_2) = \mathcal{E}(V, u_1) + \mathcal{E}(V, u_2).$$

On the other hand, since $u_1 = u_2$ on ∂U in the trace sense, also $u_1 \wedge u_2, u_1 \vee u_2 = u_1 = u_2$ on ∂U in the trace sense. So, from the minimizer assumption on u_1 and u_2, we conclude that all four terms in the above equality must be equal. Thus u_1, u_2, $u_1 \wedge u_2$, and $u_1 \vee u_2$ are all minimizers of $\mathcal{E}(V, \cdot)$ over the admissible class. In particular each one is harmonic in its positivity set. By unique continuation for harmonic functions this means that u_1 and u_2 are ordered in each connected component of $U \cap \{u_1 \vee u_2 > 0\}$. Note that by harmonicity again if $u_1 \neq u_2$ in a given connected component of $U \cap \{u_1 \vee u_2 > 0\}$ then the ordering is strict in it. $\square$

5.3.2. *Abstract left-right limits.* Next we discuss some general facts about bounded variation maps on intervals of the line $\mathbb{R}$. Suppose (Y, d_Y) is a complete metric space.

Definition 5.13. Say $f : [a, b] \to Y$ is a bounded variation map if

$$[f]_{\mathrm{BV}([a,b];Y)} := \sup_{a = t_0 \leq t_1 \leq \cdots \leq t_{N+1} = b} \sum_{j=0}^{N} d(f(t_{j+1}), f(t_j)) < +\infty.$$

On open or half open intervals define

$$[f]_{\mathrm{BV}([a,b);Y)} := \lim_{b' \nearrow b} [f]_{\mathrm{BV}([a,b'];Y)},$$

or similar for $(a, b]$ and (a, b). The limit exists by monotonicity.

Definition 5.14. Let $f : [a, b] \to Y$. For each $t \in [a, b]$, if the limits exist define the left limit $f_\ell(t)$ at $t \in (a, b]$

$$f_\ell(t) := \lim_{s \to t-} f(s)$$

and the right limit $f_r(t)$ at $t \in [a, b)$

$$f_r(t) := \lim_{s \to t+} f(s).$$

Lemma 5.15. *Let (Y, d) a complete metric space and f a bounded variation map $[a, b] \to Y$. Then f has left and right limits at each $t \in (a, b]$ (resp. $[a, b)$) and f has at most countably many jump discontinuities.*

The proof is standard, but we give a sketch for convenience.

Proof. For the directional limits note that for any sequence $s_j \nearrow t \in (a,b]$

$$\sum_j d_Y(f(s_j), f(s_{j+1})) \leq [f]_{\mathrm{BV}([a,b];Y)} < +\infty$$

and so $f(s_j)$ is Cauchy in Y. The limits must agree on different approaching sequences by a typical interlacement argument with pairs of sequences. A symmetric argument produces the right limits.

Now, first arguing for any finite set $\mathcal{T}$ of times, and then taking the supremum over all finite sets

$$\sum_{t \in [a,b]} d_Y(f_\ell(t), f_r(t)) \leq [f]_{\mathrm{BV}([a,b];Y)}.$$

In particular $f_\ell(t) = f_r(t)$ except for at most countably many times.

□

5.3.3. *Characterization of the upper and lower semicontinuous envelopes for energy solutions.* Lemma 5.15 and Lemma 5.7 together yield the existence of left and right limits of $\Omega(u(t))$, $\mathcal{J}(u(t))$, and $P(u(t))$ at all times. Our next result says that $u(t)$ also has left and right limits in the uniform metric, even though we do not necessarily establish that $t \to u(t)$ is a BV in time map into $C_c(\mathbb{R}^d)$. The left and right limits of all the previous quantities are consistent, i.e. $(P \circ u)_\ell(t) = P(u_\ell(t))$ etc. Furthermore the left and right limits of $u(t)$ satisfy certain global minimality properties for the energy plus dissipation distance.

The special structure of the problem comes in when we prove certain monotonicity properties of all the jumps, namely properties (iii)-(iv) below. This allows us to make a simple classification of the upper and lower semicontinuous envelopes of energy solutions, showing that they are energy solutions themselves. This simple jump structure will be very useful in Section 7, where we prove a certain weak version of the dynamic slope condition, and make the following proposition a central result of this section.

Semicontinuous envelopes are typically important in studying the geometric properties of interface evolution problems. They are used, for instance, in the notion of discontinuous viscosity solutions for free boundary problems, as seen in Section 4. The fact that these envelopes are themselves energy solutions is extremely useful to us in Section 7.

In service of analyzing the properties of energy solutions at jump times and simplifying repetitive notations let us introduce the set

$$\mathcal{U}(t) = \{u_\ell(t), u(t), u_r(t), u_\ell(t) \wedge u_r(t), u_\ell(t) \vee u_r(t)\} \tag{5.14}$$

which is a kind of multi-valued version of $u(t)$ indexing the values taken by $u(t)$ "near" time t as well as the upper and lower envelopes. The set $\mathcal{U}(t)$ comes with a natural partial order induced by the time variable. Since $u_\ell(t)$ is the limit from the left of $u(t)$ it is "before" all the other elements of $\mathcal{U}(t)$, and since $u_r(t)$ is the limit from the right it is "after" all the other elements of $u(t)$. Specifically we define the partial order

$$u_\ell(t) \ \trianglelefteq \ u(t),\ u_\ell(t) \wedge u_r(t),\ u_\ell(t) \vee u_r(t) \ \trianglelefteq \ u_r(t). \tag{5.15}$$

To be clear this partial ordering is purely related to the "time" of the elements of $\mathcal{U}(t)$ and is not related to the spatial ordering between the different elements of $\mathcal{U}(t)$.

The core step in the following proposition is part (ii). Essentially we show that every element of $\mathcal{U}(t)$ is a valid intermediate state for the evolution at time t. More specifically, the energy dissipation relation is still satisfied as long as the system jumps from $u_\ell(t)$ to any element of $\mathcal{U}(t)$ and then to $u_r(t)$.

Proposition 5.16. *Suppose u is an energy solution on $[0,T]$ then:*

(i) (Left and right limits exist and are consistent) At every $t \in [0,T]$, $u(t)$, $\mathcal{J}(u(t))$, $P(u(t)$, and $\Omega(u(t))$ have left and right limits and

$$\lim_{s\to_{\ell/r}t} \Omega(u(t)) = \Omega(u_{\ell/r}(t)),\ \lim_{s\to_{\ell/r}t} \mathcal{J}(u(t)) = \mathcal{J}(u_{\ell/r}(t)), \\ \textit{and} \quad \lim_{s\to_{\ell/r}t} P(u(t)) = P(u_{\ell/r}(t)) \tag{5.16}$$

(ii) (Jumps minimize energy plus dissipation) At every $t \in [0,T]$ for any $v, w \in \mathcal{U}(t)$ (from (5.14)), with $v \trianglelefteq w$ (from (5.15))

$$w \quad \textit{minimizes} \quad \mathcal{E}(v,\cdot) \quad \textit{over} \quad F(t) + H^1_0(U).$$

Note that in particular this implies all $v \in \mathcal{U}(t)$ are globally stable since $v \trianglelefteq v$.

(iii) (Jumps are monotone per component) At every $t \in [0,T]$

$$u_\ell(t) \wedge u_r(t) \le u(t) \le u_\ell(t) \vee u_r(t).$$

In particular at each $t \in [0,T]$ the upper semicontinuous and lower semicontinuous envelopes are $u^*(t) = u_r(t) \vee u_\ell(t)$ and $u_*(t) = u_r(t) \wedge u_\ell(t)$.

(iv) (Re-definition at jumps) Suppose that $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfies the property:

$$\textit{for each } t \in [0,T] \quad v(t) \in \mathcal{U}(t) \quad \textit{(from (5.14))}.$$

Then $v(t)$ is also an energy solution on $[0,T]$. In particular the upper and lower semicontinuous envelopes $u^*(t)$ and $u_*(t)$ are also energy solutions.

Besides the independent interest in understanding the time regularity properties of energetic solutions, this result plays a key role in establishing the validity of the free boundary motion law in Theorem 7.1.

Remark 5.17. Note that part (iii) does allow the possibility that the free boundaries $\partial\Omega(u_\ell(t))$ and $\partial\Omega(u_r(t))$ are distinct and monotonically ordered but touch in some nontrivial region. This is possible to occur: for example consider a small ball and a large ball merging with a large pinning interval. This would allow a portion of the free boundary opposite the "contacting area" to stay fixed as depicted in the simulation Figure 2. In other words, although strong maximum principle holds for the solutions, it does not hold for the free boundaries.

Remark 5.18. Note that from Lemma 5.11 and (iii) in each connected component V of $U \cap \{u^*(t) > 0\}$ we have $\{u^*(t,\cdot)|_V, u_*(t,\cdot)|_V\} = \{u_r(t)|_V, u_\ell(t)|_V\}$.

Remark 5.19. Note that $v(t)$ as in (iv) is always measurable since $v(t) = u(t)$ except at countably many times.

Proof of Proposition 5.16. **Part (i).** By Lemma 5.7 the map

$$t \mapsto (\mathbf{1}_{\Omega(u(t))}, \mathcal{J}(u(t)), P(u(t))) \quad \text{is in} \quad BV([0,T]; L^1(\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R})$$

and so by Lemma 5.15 the map has left and right limits in $L^1(\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}$ at each $t \in [0,T]$.

Call $\Omega_\ell(t_0)$ to be the set obtained by the L^1 left limit of $\Omega(u(t))$ at t_0. Let $t_n \nearrow t_0$. By uniform Lipschitz continuity of $u(t_n)$ there is a subsequence converging uniformly to some u_∞ which satisfies $u_\infty = F(t_0)$ on ∂U. By uniform non-degeneracy of the $u(t)$ we must have $\Omega(u_\infty) = \Omega_\ell(t_0)$. This specifies that u_∞ is the unique solution of

(5.17)
$$\Delta u_\infty = 0 \ \text{ in } \ \Omega_\ell(t_0) \cap U \ \text{ with } \ u_\infty = 0 \ \text{ on } \ \partial\Omega_\ell(t_0) \ \text{ and } \ u_\infty = F(t_0) \ \text{ on } \ \partial U.$$

Since the original sequence $t_n \nearrow t_0$ was arbitrary we find that $u(t) \to u_\infty$ uniformly as $t_n \nearrow t_0$. The argument is the same for right limits.

Thus $u(t)$ has left and right limits $u_{\ell/r}(t)$ in the uniform metric at each time $t \in [0,T]$. Next we need to show the consistency properties (5.16). We already showed that $\lim_{s\to_{\ell/r}t} \Omega(u(t)) = \Omega(u_{\ell/r}(t))$ using uniform non-degeneracy in the previous paragraph. Recalling that $P(u(t)) = F(t)^{-1}\mathcal{D}(u(t))$, to conclude we just need to check the consistency of limits for the Dirichlet energy $\mathcal{D}(u(t))$

$$\lim_{s\to_{\ell/r}t} \int_U |\nabla u(t)|^2 \, dx = \int_U |\nabla u_{\ell/r}(t)|^2 \, dx. \tag{5.18}$$

We claim that $\nabla u(s)$ converges locally uniformly on $\Omega(u_{\ell/r}(t))$ to $\nabla u_{\ell/r}(t)$ as $s \nearrow t$ (resp. $s \searrow t$). Along with the uniform boundedness of $\nabla u(t)$ the dominated convergence theorem gives (5.18).

As usual we handle the left limit case and the right limit case is similar. Let K be a compact subset of $\Omega(u_\ell(t))$. By the uniform convergence $u(s) \to u_\ell(t)$ as $s \nearrow t$ and by uniform non-degeneracy

$$\inf_K u(s) \geq c d(K, \Omega(u_\ell(t))^{\complement})$$

and so by uniform Lipschitz continuity

$$d(K, \Omega(u(s))^{\complement}) \geq c d(K, \Omega(u_\ell(t))^{\complement})$$

for $s < t$ sufficiently close to t. Thus the $\nabla u(s)$ are uniformly bounded and uniformly continuous on K for $s < t$ sufficiently close to t. Typical arbitrary subsequence arguments show that $\nabla u(s)$ converges uniformly on K to $\nabla u_\ell(t)$ as $s \nearrow t$.

Part (ii). We introduce the extraneous notation $u_0(t) = u(t)$ in order to prove minimizer properties for $u_\ell/u_0/u_r$ at once.

First we check that each of $u_{\ell/0/r}(t)$ minimize their own respective $\mathcal{E}(\Omega(u_{\ell/0/r}(t)), \cdot)$ over $F(t) + H^1_0(U)$. For $u_0(t)$ this is just the global stability property of an energy solution. For $u_\ell(t)$, by the global stability property for any $t_- < t$ we know

$$\mathcal{J}(u(t_-)) \leq J\left(\frac{F(t_-)}{F(t)}v\right) + \operatorname{Diss}(u(t_-), v) \ \text{ for all } \ v \in F(t) + H^1_0(U).$$

Taking the limit as $t_- \nearrow t$, using time continuity of F, and applying part (i)

$$\mathcal{J}(u_\ell(t)) \leq \mathcal{J}(v) + \operatorname{Diss}(u_\ell(t), v) \ \text{ for all } \ v \in F(t) + H^1_0(U).$$

The proof that $u_r(t)$ minimizes $\mathcal{E}(u_r(t), \cdot)$ is similar, applying global stability of $u(t_+)$ and taking a limit $t_+ \searrow t$.

Next we check that $u_{0/r}$ minimize $\mathcal{E}(u_\ell(t), \cdot)$ over $F(t) + H^1_0(U)$. Apply the dissipation relation with times $t_- \leq t \leq t_+$

$$\mathcal{J}(u(t_-)) + \int_{t_-}^{t_+} 2\dot{F}(s)P(s) \, ds \geq \operatorname{Diss}(u(t_-), u(t_+)) + \mathcal{J}(u(t_+)).$$

Taking the limit as $t_- \nearrow t$ and $t_+ \searrow t$ and using the left/right continuity of $\mathcal{J}(u(\cdot))$ and $\Omega(\cdot)$ established in part (i)

$$\mathcal{J}(u_\ell(t)) \geq \mathcal{J}(u_r(t)) + \mathrm{Diss}(u_\ell(t), u_r(t))$$

or in the case $t_+ = t$ and $t_- \nearrow t$

$$\mathcal{J}(u_\ell(t)) \geq \mathcal{J}(u(t)) + \mathrm{Diss}(u_\ell(t), u(t))$$

Since we already established that $u_\ell(t)$ is always a minimizer of $\mathcal{E}(u_\ell(t), \cdot)$ also $u_{0/r}(t)$ must be minimizers. Finally in the case $t_- = t$ and $t_+ \searrow t$ we obtain similarly

$$\mathcal{J}(u(t)) \geq \mathcal{J}(u_r(t)) + \mathrm{Diss}(u(t), u_r(t))$$

which gives that $u_r(t)$ minimizes $\mathcal{E}(u(t), \cdot)$.

Lastly we consider $u_\ell(t) \wedge u_r(t)$ and $u_\ell(t) \vee u_r(t)$. Since t is fixed for the remainder of this part of the proof we will write $u_\ell = u_\ell(t)$ and $u_r = u_r(t)$ to simplify expressions.

First of all, since u_ℓ and u_r minimize $\mathcal{E}(u_\ell, \cdot)$ over $F(t) + H_0^1(U)$, so also, by (5.13), $u_\ell \wedge u_r$ and $u_\ell \vee u_r$ both minimize $\mathcal{E}(u_\ell, \cdot)$ over $F(t) + H_0^1(U)$.

Then for any $v \in F(t) + H_0^1(U)$

$$\mathcal{E}(u_\ell, u_\ell \wedge u_r) \leq \mathcal{E}(u_\ell, v)$$

or

$$\begin{aligned}
\mathcal{J}(u_\ell \wedge u_r) &\leq \mathcal{J}(v) + \mathrm{Diss}(u_\ell, v) - \mathrm{Diss}(u_\ell, u_\ell \wedge u_r) \\
&\leq \mathcal{J}(v) + \mathrm{Diss}(u_\ell \wedge u_r, v) \\
&= \mathcal{E}(u_\ell \wedge u_r, v)
\end{aligned}$$

by the dissipation distance triangle inequality Lemma A.3 for the last inequality. Thus $u_\ell \wedge u_r$ is globally stable. A similar argument applies to $u_\ell \vee u_r$.

Finally we need to argue that

$$u_r \quad \text{minimizes } \mathcal{E}(u_\ell \wedge u_r, \cdot) \text{ and } \mathcal{E}(u_\ell \vee u_r, \cdot) \text{ over } F(t) + H_0^1(U).$$

We just argue for $\mathcal{E}(u_\ell \wedge u_r, \cdot)$, the other case is similar,

$$\begin{aligned}
\mathcal{E}(u_\ell \wedge u_r, v) &= \mathcal{E}(u_\ell, v) + \mathrm{Diss}(u_\ell \wedge u_r, v) - \mathrm{Diss}(u_\ell, v) \\
&\geq \mathcal{E}(u_\ell, u_r) + \mathrm{Diss}(u_\ell \wedge u_r, v) - \mathrm{Diss}(u_\ell, v) \\
&= \mathcal{J}(u_r) + \mathrm{Diss}(u_\ell, u_r) + \mathrm{Diss}(u_\ell \wedge u_r, v) - \mathrm{Diss}(u_\ell, v) \\
&= \mathcal{J}(u_r) + \mathrm{Diss}(u_\ell \wedge u_r, v) + \mathrm{Diss}(v, u_r) \\
&\qquad + [\mathrm{Diss}(u_\ell, u_r) - \mathrm{Diss}(u_\ell, v) - \mathrm{Diss}(v, u_r)] \\
&= \mathcal{J}(u_r) + \mathrm{Diss}(u_\ell \wedge u_r, u_r) \\
&\qquad + (\mu_- + \mu_+)[|\Omega(v) \setminus \Omega((u_\ell \wedge u_r) \vee u_r)| + |\Omega((u_\ell \wedge u_r) \wedge u_r) \setminus \Omega(v)|] \\
&\qquad - (\mu_- + \mu_+)[|\Omega(v) \setminus \Omega(u_\ell \vee u_r)| + |\Omega(u_\ell \wedge u_r) \setminus \Omega(v)|] \\
&= \mathcal{E}(u_\ell \wedge u_r, u_r).
\end{aligned}$$

where we have used that u_r minimizes $\mathcal{E}(u_\ell, \cdot)$, then the sharp triangle inequality Lemma A.3, and finally that $(u_\ell \wedge u_r) \wedge u_r = u_\ell \wedge u_r$ and $(u_\ell \wedge u_r) \vee u_r = u_\ell \vee u_r$.

Part (iii). Since t is fixed in this part of the proof we write $u_{\ell/0/r}(t) = u_{\ell/0/r}$ dropping the t dependence. By global stability and part (ii) respectively we know that u_ℓ and u_r both minimize $\mathcal{E}(u_\ell, \cdot)$. Furthermore Lemma 5.11 yields that u_ℓ and u_r are ordered in each connected component of $\Omega(u_\ell \vee u_r)$.

Next we show that $u_\ell \wedge u_r \leq u \leq u_\ell \vee u_r$. We use the multiple minimizations in part (ii) to show an "equality in triangle inequality" which implies that u must be in between u_ℓ and u_r. Intuitively speaking an intermediate jump which crosses either u_ℓ or u_r would just incur extra dissipation cost to return to u_r.

We compute using part (ii)

$$\begin{aligned}\mathcal{J}(u_r) + \mathrm{Diss}(u_\ell, u_r) &= \mathcal{J}(u) + \mathrm{Diss}(u_\ell, u)\\ &= \mathcal{J}(u_r) + \mathrm{Diss}(u, u_r) + \mathrm{Diss}(u_\ell, u).\end{aligned}$$

More specifically the first equality is since u_r and u both minimize $\mathcal{E}(u_\ell, \cdot)$, the second equality is since u and u_r both minimize $\mathcal{E}(u, \cdot)$. Simplifying we find the equality

$$\mathrm{Diss}(u_\ell, u) + \mathrm{Diss}(u, u_r) - \mathrm{Diss}(u_\ell, u_r) = 0. \tag{5.19}$$

By the sharp triangle inequality Lemma A.3

$$\mathrm{Diss}(u_\ell, u) + \mathrm{Diss}(u, u_r) - \mathrm{Diss}(u_\ell, u_r) = (\mu_- + \mu_+)\Big[|\Omega(u_\ell \wedge u_r) \setminus \Omega(u)| + |\Omega(u) \setminus \Omega(u_\ell \vee u_r)|\Big].$$

Together with (5.19), above equality shows that $\Omega(u_\ell \wedge u_r) \subset \Omega(u) \subset \Omega(u_\ell \vee u_r)$. Since $u_\ell \wedge u_r$, u, and $u_\ell \vee u_r$ are harmonic in their respective positivity sets by Lemma 5.11, then the previous set ordering and maximum principle implies $u_\ell \wedge u_r \leq u \leq u_\ell \vee u_r$.

Part (iv). Since $u_\ell(t) = u_r(t) = u(t) = F(t)$ on ∂U the boundary condition is satisfied by $v(t)$. By (ii) any $v(t) \in \mathcal{U}(t)$ is globally stable.

Finally we need to check the dissipation relation (1.6) for v. We do this in two steps, the dissipation on the open interval (t_0, t_1) plus the dissipation at the endpoints.

Fix $t_0 < t_1$ and we claim that

$$\mathcal{J}(u_r(t_0)) - \mathcal{J}(u_\ell(t_1)) + \int_{t_0}^{t_1} 2\dot{F}(s)P(s)\, ds \geq \mathrm{Diss}(u_r(t_0), u_\ell(t_1)) \tag{5.20}$$

To prove this apply the dissipation relation for $u(t)$ with sequences $t_{0,k}$ and $t_{1,k}$ with $t_{0,k} \searrow t_0$ and $t_{1,k} \nearrow t_1$

$$\mathcal{J}(u(t_{0,k})) - \mathcal{J}(u(t_{1,k})) + \int_{t_{0,k}}^{t_{1,k}} 2\dot{F}(s)P(s)\, ds \geq \mathrm{Diss}(u(t_{0,k}), u(t_{1,k})).$$

Sending $k \to \infty$ and using the continuities from (i) shows (5.20).

Now consider the endpoint dissipations, by (ii),

$$\mathcal{J}(u_r(t_0)) + \mathrm{Diss}(v(t_0), u_r(t_0)) = \mathcal{J}(v(t_0)) \ \text{ and } \ \mathcal{J}(v(t_1)) + \mathrm{Diss}(u_\ell(t_1), v(t_1)) = \mathcal{J}(u_\ell(t_1))$$

since every $v(t_0) \trianglelefteq u_r(t_0)$ for every $v(t_0) \in \mathcal{U}(t_0)$ and $u_\ell(t_1) \trianglelefteq v(t_1)$ for every $v(t_1) \in \mathcal{U}(t_1)$ in the temporal partial ordering (defined in (5.15)).

Adding these to (5.20) gives

$$\begin{aligned}&\mathcal{J}(v(t_0)) - \mathcal{J}(v(t_1)) + \int_{t_0}^{t_1} 2\dot{F}(s)P(s)\, ds\\ &\quad\geq \mathrm{Diss}(v(t_0), u_r(t_0)) + \mathrm{Diss}(u_r(t_0), u_\ell(t_1)) + \mathrm{Diss}(u_\ell(t_1), v(t_1))\\ &\quad\geq \mathrm{Diss}(v(t_0), v(t_1))\end{aligned}$$

by applying the triangle inequality Lemma A.3 in the last step. This completes the proof. $\square$

6. Limit of the minimizing movement scheme

In this section we will study the minimizing movements scheme introduced in (1.7). We recall that scheme here for convenience:

$$u_\delta^k \in \operatorname{argmin}\left\{\mathcal{J}(w) + \operatorname{Diss}(u_\delta^{k-1}, w) : w \in F(k\delta) + H_0^1(U)\right\}.$$

Then interpolate discontinuously to define, for all $t \in [0,T]$,

$$u_\delta(t) := u_\delta^k \quad \text{and} \quad F_\delta(t) = F(k\delta) \quad \text{if} \quad t \in [k\delta, (k+1)\delta).$$

We will analyze the behavior of the minimizing movements scheme as $\delta \to 0$.

First, in Section 6.1, we apply established ideas [1, 27] to show that, up to a subsequence, the u_δ converge pointwise in time to an energy solution (E). Our argument will be quite similar to that in Alberti and DeSimone [1, Section 4] which was itself inspired by Mainik and Mielke [27]. The conclusion of these arguments is that, up to a subsequence, the $u_\delta(t)$ and $\Omega(u_\delta(t))$ converge pointwise *everywhere* in time in uniform / Hausdorff distance norm in space to a (global) energetic solution.

Then, in Section 6.2 below, we will show that the approximate solutions generated by this scheme will also satisfy the dynamic slope condition (M) in the viscosity sense. In the star-shaped case we will be able to apply the comparison principle proved in Section 4 to show convergence to the unique obstacle solution (O), see Theorem 6.6. As a corollary we obtain that the obstacle solution (O) is an energy solution.

These results will complete the proof of our first main theorem in the introduction, Theorem 1.5.

Remark 6.1. The very recent result of Ferreri and Velichkov [22] implies that $C^{1,\beta}$ regularity is propagated in the discrete scheme (1.7). However, at least with a naive application of their result, the constants would blow-up as $\delta \to 0$ since the regularity theorem needs to be applied $O(\frac{1}{\delta})$ times in order to get to a nontrivial positive t in the limit. In the star-shaped case we go around this difficulty due to the equivalence with the (O) evolution. This equivalence allows us to apply the regularity theorem only finitely many times, at each monotonicity change, as in Lemma 4.22.

6.1. **Pointwise in time convergence to an energy solution.** The goal of this section is to analyze the limits of the minimizing movements scheme by energetic methods. We start by proving uniform (in $\delta > 0$) BV in time bounds for the minimizing movements scheme via a discrete Grönwall type argument similar to Lemma 5.7. This establishes the necessary compactness to show that, up to a subsequence, the $u_\delta(t)$ and $\Omega(u_\delta(t))$ converge at *every* time in uniform / Hausdorff distance norm to a (global) energetic solution. The convergence at *every* time, which follows from an application of Helly's selection theorem, is a key detail to establish the energy solution property.

Theorem 6.2. *The $u_\delta(t)$ generated by the minimizing movements scheme satisfy the following:*

(i) The states $u_\delta(t)$ are uniformly Lipschitz, $\chi_{\Omega_\delta(t)}$ and $\mathcal{J}(u_\delta)$ are uniformly bounded in $BV([0,T];L^1(U))$ and $BV([0,T];\mathbb{R})$ respectively.

(ii) There is a subsequence $\delta_k \to 0$ and $u : [0,T] \to F(t) + H_0^1(U)$ so that for every $t \in [0,T]$

$$\|u_{\delta_k}(t) - u(t)\|_{L^\infty(U)} \ \textit{and} \ \ d_H(\Omega(u_{\delta_k}(t)), \Omega(u(t))) \to 0 \quad \textit{as } \delta_k \to 0.$$

(iii) Any such subsequential limit $u(t)$ is an energy solution.

Remark 6.3. By Theorem 6.2 any minimizing movements solution, Definition 1.2, is indeed an energy solution of (E).

The proof will use many of the same ideas that were developed in Section 5 in the continuous time case, so we will refer to the relevant arguments above in many places to significantly shorten the presentation.

Proof. **Part (i).** A discrete version of the energy dissipation inequality will play a central role in the proof. We claim that for each $k \geq 1$

$$\mathrm{Diss}(u_\delta^{k-1}, u_\delta^k) \leq \int_{(k-1)\delta}^{k\delta} 2(1+g_\delta(t))\dot{F}(t)P_\delta(t)\, dt + \mathcal{J}(u_\delta^{k-1}) - \mathcal{J}(u_\delta^k) \tag{6.1}$$

where the error term in the integrand has

$$|g_\delta(t)| \leq \exp(\|(\log F)'\|_\infty \delta) - 1 \to 0 \quad \text{uniformly as } \delta \to 0, \tag{6.2}$$

and we used the notation $P_\delta(t) := \int_{\partial U} \frac{\partial u_\delta^{[t/\delta]}}{\partial n} dS$.

To prove this by using the minimizing scheme we build a comparable version of u_δ^{k-1} to u_δ^k, by defining $\tilde{u} := \frac{F(k\delta)}{F((k-1)\delta)} u_\delta^{k-1}$. Then we have

$$\mathrm{Diss}(u_\delta^{k-1}, u_\delta^k) + \mathcal{J}(u_\delta^k) \leq \mathrm{Diss}(u_\delta^{k-1}, \tilde{u}) + \mathcal{J}(\tilde{u}) = \mathcal{J}(\tilde{u}) = \mathcal{J}(\tilde{u}) - \mathcal{J}(u_\delta^{k-1}) + \mathcal{J}(u_\delta^{k-1}).$$

By integration by parts we can rewrite

$$\begin{aligned}\mathcal{J}(\tilde{u}) - \mathcal{J}(u_\delta^{k-1}) &= \frac{F_k^2 - F_{k-1}^2}{F_{k-1}} \int_{\partial U} \frac{\partial u_\delta^{k-1}}{\partial n} dS \\ &= \int_{(k-1)\delta}^{k\delta} 2\frac{F(t)}{F_{k-1}} \dot{F}(t) P_\delta(t)\, dt.\end{aligned}$$

Then define $g_\delta(t) := \frac{F(t)}{F_{k-1}} - 1$ for $[t] = k-1$ which satisfies (6.2) by fundamental theorem of calculus and the inequality $1 - e^{-k} \leq e^k - 1$ for $k > 0$. That completes our proof of (6.1).

Next we sum up the one-step dissipation inequality (6.1) to get, for any $0 \leq t_0 \leq t_1 \leq T$,

$$\overline{\mathrm{Diss}}(u_\delta(t); [t_0, t_1]) \leq \int_{t_0}^{t_1} 2(1+g_\delta(t))\dot{F}(t)P_\delta(t) dt + \mathcal{J}(u_\delta(t_1)) - \mathcal{J}(u_\delta(t_0)) + C\delta \tag{6.3}$$

where $C\delta$ accounts for the error from t_0 and t_1 not necessarily being integer multiples of δ and we are using the uniform Lipschitz bound of $u_\delta(t)$ from Lemma 5.5.

From (6.3) we can derive uniform $BV([0,T]; L^1(U) \times \mathbb{R} \times \mathbb{R})$ bounds on the map $t \mapsto (\chi_{\Omega_\delta}(t), P_\delta(t), \mathcal{J}_\delta(t))$ by the same arguments as in Lemma 5.7.

Part (ii). Due to the above compactness, Helly's selection principle [1, Theorem 5.1] yields that, along a subsequence,

$$(\chi_{\Omega_\delta}(t), P_\delta(t), \mathcal{J}_\delta(t)) \to (\chi_{\Omega(t)}, P(t), \mathcal{J}(t)) \quad \text{in} \quad L^1(U) \times \mathbb{R} \times \mathbb{R} \quad \text{for } \textit{every } t \in [0,T].$$

Then arguing as in the proof of part (i) of Proposition 5.16 we can also show that $u_\delta(t) \to u(t)$ uniformly in U for each $t \in [0,T]$ and we can show the consistency of the limits

$$\Omega(t) = \Omega(u(t)), \; \mathcal{J}(t) = \mathcal{J}(u(t)), \quad \text{and} \quad P(t) = P(u(t)).$$

Furthermore the convergence of $\Omega_\delta(t) \to \Omega(t)$ can be upgraded to Hausdorff distance convergence using the uniform convergence of the $u_\delta(t) \to u(t)$ and the uniform non-degeneracy of the $u_\delta(t)$.

Part (iii). Now let us show that $u(t)$ is an energy solution. To show the stability property, observe that for any $v \in u(t) + H^1_0(U)$ we can take $k = [t/\delta]$ and compare $u_\delta(t) = u^k_\delta$ with $\tilde{v} := \frac{F(\delta k)}{F(t)} v(t)$, so that

$$\mathrm{Diss}(u^{k-1}_\delta, u^k_\delta) + \mathcal{J}(u^k_\delta) \leq \mathrm{Diss}(u^{k-1}_\delta, \tilde{v}) + \mathcal{J}(\tilde{v}).$$

Note that, due to the possibility of a jump in the limit we don't necessarily know that $\mathrm{Diss}(u^{k-1}_\delta, u^k_\delta) \to 0$. So to get everything in terms of $u_\delta(t) = u^k_\delta$ we apply the triangle inequality Lemma A.3

$$\begin{aligned}\mathcal{J}(u_\delta(t)) = \mathcal{J}(u^k_\delta) &\leq \mathrm{Diss}(u^{k-1}_\delta, \tilde{v}) - \mathrm{Diss}(u^{k-1}_\delta, u^k_\delta) + \mathcal{J}(\tilde{v}) \\ &\leq \mathrm{Diss}(u^k_\delta, \tilde{v}) + \mathcal{J}(\tilde{v}) \\ &= \mathrm{Diss}(u_\delta(t), \tilde{v}) + \mathcal{J}(\tilde{v})\end{aligned}$$

Now sending $\delta \to 0$ along the convergent subsequence yields the desired stability inequality.

Last we show the energy dissipation inequality. For any $0 \leq t_0 < t_1 \leq T$ we can apply (6.3) to find

$$\mathrm{Diss}(u_\delta(t_0), u_\delta(t_1)) \leq \int_{t_0}^{t_1} 2(1 + g_\delta(t))\dot{F}(t) P_\delta(t) dt + \mathcal{J}(u_\delta(t_1)) - \mathcal{J}(u_\delta(t_0)) + C\delta.$$

Using the consistency results from part (ii) above and sending $\delta \to 0$ gives the dissipation inequality for u. The convergence of the left hand side just follows from the L^1 convergence of the indicator functions, and the convergence of the integral is by dominated convergence theorem. □

6.2. **Comparison properties of the minimizing movements scheme.** In this section we discuss the comparison properties of the minimizing movement scheme and its $\delta \to 0$ limit.

Lemma 6.4. *The $u_\delta(t)$ are viscosity solutions of* (M) *on $[0, T]$ in the sense of Definition 1.4.*

Remark 6.5. The $u_\delta(t)$ are also viscosity solutions of (M) in the "standard" sense of touching from above and below by C^1 test functions in space-time. This notion, unlike the notion in Definition 1.4 which defines positive velocity based on cones, behaves well with respect to the limit $\delta \to 0$. One can apply the standard method of upper and lower half-relaxed limits, of Barles and Perthame [3], to show that $\overline{u}^*$ and $\underline{u}_*$ are respectively sub and supersolutions of (M) in the standard sense.

We should remark that the upper and lower-half relaxed limits $\overline{u}^*$ and $\underline{u}_*$ are not necessarily the same as the upper and lower semicontinuous envelopes $u^*(t)$ and $u_*(t)$ of the pointwise limit $u(t)$ constructed in Theorem 6.2. It would be interesting to know whether the pointwise in time (subsequential) limits $u(t)$ are also viscosity solutions of (M) in general either in the standard sense or in the cone sense of Definition 1.4. Below in Theorem 6.6 we will show that it is so in the star-shaped case.

Proof. We only check the subsolution condition at the free boundary since the remaining cases are standard or symmetrical.

Suppose that $\Omega(u_\delta^*(t))$ has positive normal velocity $V_n(t_0, x_0) > 0$ at some point $x_0 \in \partial\Omega(u_\delta^*(t_0))$ in the sense of Definition 4.9, i.e.

$$\{x : \ |x - x_0| \leq c(t - t_0)\} \subset \Omega(u_\delta^*(t))^{\complement} \ \text{ for } \ t_0 - r_0 \leq t < t_0.$$

Since u_δ is constant on open intervals of the form $(k\delta, (k+1)\delta)$ this implies that $t_0 = k\delta$ for an integer k, that $u_\delta^*(t_0) = u_\delta^k$, and that $u_\delta^*(t) = u_\delta^{k-1}$ for $t_0 - \delta \leq t < t_0$. Since u_δ^k is a minimizer of

$$\mathcal{E}(u_\delta^{k-1}, v) = \mathcal{J}(v) + \mathrm{Diss}(u_\delta^{k-1}, v) \ \text{ over } \ v \in F(k\delta) + H_0^1(U)$$

and we have just established that $x_0 \in \mathbb{R}^d \setminus \overline{\Omega(u_\delta^{k-1})}$ we can apply Lemma 5.2 and Lemma 5.3 to conclude that

$$|\nabla u(t_0, x_0)|^2 \geq 1 + \mu_+ \text{ in the viscosity sense.}$$

□

Now we can apply the previous viscosity solution property in combination with the comparison theorem in Section 4 to show that, in the star-shaped case, the limit of the minimizing movements scheme is the same as the solution of the obstacle evolution (O). The convergence is in fact quantitative.

Theorem 6.6. *Suppose that Ω_0 is a $C^{1,\alpha}$ strongly star-shaped region, $K = \mathbb{R}^d \backslash U$ is compact and strongly star-shaped, F is positive Lipschitz continuous, and F changes monotonicity at most finitely many times on $[0, T]$. Call $v(t)$ the obstacle solution* (O) *and $u_\delta(t)$ to be a solution of the time incremental scheme (1.7) on $[0, T] \times U$. Then*

$$\|u_\delta(t) - v(t)\|_{L^\infty(U)} \leq C(\mu_\pm, d)(\exp(\|(\log F)'\|_\infty \delta) - 1)\|F\|_\infty \to 0 \quad \text{as} \quad \delta \to 0.$$

Thus the viscosity solution of (O) *is the unique minimizing movements solution of the energetic evolution (Definition 1.2), in particular any minimizing movements solution, Definition 1.2, is also a motion-law viscosity solution* (M)*.*

Let us clarify the conclusions of this statement, since several threads of the paper have suddenly come together. We have shown the following:

- In the star-shaped setting the obstacle solution (O) is an energy solution.
- In the star-shaped setting there is only one minimizing movements energy solution (Definition 1.2) and it is also a motion-law viscosity solution (M).
- In the star-shaped setting all solutions of the discrete scheme converge uniformly to the obstacle solution (O) with quantitative rate.

Proof. We prove the convergence by comparison principle Proposition 4.26 with $v(t)$ the (O) solution which, by Corollary 4.25, is in $C_t^{0,1} L_x^\infty \cap L_t^\infty C_x^{1,\beta}$ on $[0, T] \times U$.

Let $M(\delta) = \sup_t \sup_{h \in [0,\delta]} F(t - h)/F(t)$, note that $M(\delta) \leq \exp(\|(\log F)'\|_\infty \delta)$. Then

$$\varphi_\delta(t, x) := M(\delta) v(t, M(\delta)^{-1} x)$$

is a $C_t^{0,1} L_x^\infty \cap L_t^\infty C_x^{1,\beta}$ supersolution of (M) on $[0, T]$ with star-shaped level sets and so the comparison principle Proposition 4.26 implies that $u_\delta \leq \varphi_\delta$ on $[0, h] \times U$. Note that we have chosen $M(\delta) \geq 1$ so that for $x \in K$ since K star-shaped also $M(\delta)^{-1} x \in K$ and

$$\varphi_\delta(t, x) = M(\delta) v(t, M(\delta)^{-1} x) = M(\delta) F(t) \geq \sup_{h \in [0,\delta]} F(t - h) \geq F_\delta(t)$$

By uniform Lipschitz regularity of $v(t)$

$$\begin{aligned} u_\delta(t,x) &\leq \varphi_\delta(t,x) \\ &\leq M(\delta)v(t,x) + M(\delta)\|\nabla v(t)\|_\infty (M(\delta)^{-1}-1)\sup_{x\in\Omega(v(t))}|x| \\ &\leq v(t,x) + C(M(\delta)-1)\|F\|_\infty \end{aligned}$$

Note that $\Omega(v(t)) \subset B_{CF(t)}(0)$ by uniform non-degeneracy of $v(t)$ and the upper bound $v(t) \leq F(t)$.

This proves the upper bound, the lower bound is similar comparing with an inward dilation of size $m(\delta) = \inf_t \inf_{h\in[0,\delta]} F(t-h)/F(t)$ instead.

Furthermore, applying Theorem 6.2, we also know that, along a subsequence, $u_\delta(t) \to u(t)$ pointwise in time and uniformly in space and $u(t)$ is an energy solution. By uniqueness of pointwise limits $u \equiv v$ and so the solution of (O) is an energy solution. □

7. The dynamic slope condition for energy solutions

In this section we show that the energy dissipation balance law implies a weak notion of the dynamic slope condition (1.5)

$$|\nabla u|^2 = 1 \pm \mu_\pm \quad \text{if} \quad \pm V_n(\Omega(t),x) > 0.$$

The notion of weak solution is geometric measure theoretic, although there is some viscosity solution aspect as well since there is a subsolution and a supersolution piece which are expressed in terms of sub and superdifferentials. This will complete the proof of Theorem 1.6.

Note that in Section 6 we showed, at least in the star-shaped case, that all minimizing movements solutions are viscosity solutions of the motion law (M). However, this proof was based on the structure provided by the time incremental scheme. Even in the star-shaped case, we do not know uniqueness of energy solutions or whether all energy solutions can be generated by this scheme. Thus we must prove some notion of the dynamic slope condition (1.5) directly from the energy solution property (E).

Recall by Proposition 5.16 part (iv) if u is an arbitrary energy solution on $U \times [0,T]$ then the envelopes u^* and u_* are also energy solutions and have a simple representation in terms of the left and right limits at each time $u_*(t) = u_\ell(t) \wedge u_r(t)$ and $u^*(t) = u_\ell(t) \vee u_r(t)$.

The main theorem of this section is that if $u(t)$ is an energy solution then $u^*(t)$ and $u_*(t)$ are respectively (up to sets of surface measure zero) a motion law viscosity subsolution and supersolution.

For the statement we will use the notions of positive and negative velocity $V_n^\pm(t,x)$, based on inner and outer touching space-time cones given in Section 4, and the notions of sub and superdifferential. Specifically refer back to Definition 4.4, Definition 4.3, Definition 4.9, and Definition 4.10.

Theorem 7.1. *Suppose that u is an upper semicontinuous energy solution (E) on $U \times [0,T]$. Then for every $t \in (0,T]$ the set of points where the outward motion law fails*

$$\Gamma^+(u,t) := \{x \in \partial\Omega(t) : \ V_n(t,x) > 0, \ |p|^2 < 1 + \mu_+ \ \textit{ for some } \ p \in D_+u(t,x)\}$$

has $\mathcal{H}^{d-1}$ measure zero.

Similarly, if u is an lower semicontinuous energy solution (E) *on $U \times [0,T]$, then for every $t \in (0,T]$ the set of points where the inward motion law fails*

$$\Gamma^-(u,t) := \{x \in \partial\Omega(t) : \ V_n(t,x) < 0, \ |p|^2 > 1 - \mu_- \ \text{ for some } \ p \in D_- u(t,x)\}$$

has $\mathcal{H}^{d-1}$ measure zero.

We will only prove the result for Γ^+. The proof for Γ^- is analogous.

Remark 7.2. The sub and superdifferential may both be empty at some free boundary points. However, for energetic solutions, we can follow standard blow-up arguments (see [6, Section 3.3]) to show that the sub and super-differential are both non-trivial $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega(u(t))$.

Recall that $\partial\Omega(u(t))$ is a finite perimeter set and $\partial\Omega(u(t))$ has finite $\mathcal{H}^{d-1}$ measure (Lemma 5.5) for each $t > 0$. Therefore the reduced boundary $\partial_*\{u(t) > 0\}$ has full $\mathcal{H}^{d-1}$ measure on $\partial\{u(t) > 0\}$.

The sub and superdifferentials are both non-trivial at every point of the reduced boundary. More precisely, at $x_0 \in \partial_*\{u > 0\}$ with measure theoretic normal n_0 consider the blow-up sequence

$$u_r(x) = \frac{u(x_0 + rx)}{r}$$

which are uniformly bounded, uniformly non-degenerate, and uniformly Lipschitz. All subsequential blow up limits (not necessarily unique) must then be viscosity solutions of

$$\Delta v = 0 \ \text{ in } \ \{x \cdot n_0 > 0\} \ \text{ and } \ 1 - \mu_- \leq |\nabla v|^2 \leq 1 + \mu_+ \ \text{ on } \ \partial\{x \cdot n_0 > 0\}.$$

From this one can conclude that for any (subsequential, locally uniform) blow up limit

$$(1-\mu_-)^{1/2}(x \cdot n_0)_+ \leq v(x) \leq (1+\mu_+)^{1/2}(n_0 \cdot x)_+$$

and this implies that at least

$$(1+\mu_+)^{1/2} n_0 \in D_+ u(x_0) \ \text{ and } \ (1-\mu_-)^{1/2} n_0 \in D_- u(x_0),$$

i.e. the sub and superdifferential are nontrivial on the measure theoretic reduced boundary.

Before we proceed to the details let us give a description of the proof, which also explains why we can only prove the solution condition in an almost everywhere sense.

We start by presenting the formal argument, valid in the case that everything is C^1 in space and time. Computing the time derivative directly and integrating by parts

$$\frac{d}{dt}\mathcal{J}(u(t)) = \int_{\partial\Omega(t)} (1 - |\nabla u|^2) V_n \, dS + 2\dot{F}(t)P(t)$$

but also differentiating the energy dissipation balance from Lemma 5.10 yields

$$\frac{d}{dt}\mathcal{J}(u(t)) = 2\dot{F}(t)P(t) - \int_{\partial\Omega(t)} \mu_+(n)(V_n)_+ + \mu_-(n)(V_n)_- \, dS.$$

Combining the above two identities we find

$$\int_{\partial\Omega(t)} (1 + \mu_+(n) - |\nabla u|^2)(V_n)_+ + (|\nabla u|^2 - 1 + \mu_-(n))(V_n)_- \, dS = 0. \tag{7.1}$$

Both terms in the above integral are non-negative (Corollary 5.4), and so they must actually be zero pointwise.

Of course we cannot exactly use the formal argument. Even without jumps it is tricky to justify taking a time derivative. Instead we need to make a similar kind of energy argument with discrete differences. Furthermore, even if we could justify the identity (7.1), without continuity of ∇u we could only derive the slope condition in the surface measure a.e. sense. Recall that we do not know C^1 regularity of general energy solutions. The regularity theory of Section 3 applies to viscosity solutions and we are trying to prove a viscosity solution type property. This is why we can only achieve the dynamic slope condition in the almost everywhere sense.

Although we do need to deal carefully with the jumps, the situation is actually better when the free boundary jumps: in this case we can guarantee that slope condition is satisfied pointwise *everywhere*.

Corollary 7.3. *Let $F > 0$ and suppose that $u \geq u_\ell$ (or $u \leq u_\ell$) is a minimizer of*

$$\mathcal{E}(u_\ell, u) = \mathcal{J}(u) + \operatorname{Diss}(u_\ell, u) \quad \text{over} \quad F + H^1_0(U). \tag{7.2}$$

Then for any $x_0 \in \partial\Omega(u) \setminus \overline{\Omega(u_\ell)}$ (or $x_0 \in \partial\Omega(u) \cap \Omega(u_\ell)$) then

$$|p|^2 \geq (1 + \mu_+) \quad \text{for all} \quad p \in D^+u(x_0) \quad (\text{or } |p|^2 \leq (1 - \mu^-) \quad \text{for all} \quad p \in D^-u(x_0)).$$

Proof. Apply Lemma 5.2 and Lemma 5.3. □

In order to make the main ideas of the proof of Theorem 7.1 more clear we will state in a separate lemma the construction of the energy competitor.

First we explain the idea of the energy competitor and then give a precise statement in Lemma 7.4 below. Suppose that the advancing part of the dynamic slope condition fails at some time t_0 on a set of positive $\mathcal{H}^{d-1}$ measure. Then for times $t_{-1} = t_0 - \delta$ sufficiently close to t_0, we can perform an inward perturbation of $u(t_0)$ in the normal direction in a region with measure *at least* $O(\delta)$, and without crossing $u(t_{-1})$. This perturbation will also have slope strictly smaller than the pinning interval endpoint value $(1+\mu_+)^{1/2}$ in the perturbation region. This allows us to estimate the change in the Dirichlet energy.

Lemma 7.4. *Let u be upper semi-continuous on $[0, T]$, harmonic in $\Omega(u(t))$ at each time, and satisfying the conclusions of Lemma 5.5 (e.g. u an upper semicontinuous energy solution).*

Suppose for some $t_0 \in (0, T]$

$$\mathcal{H}^{d-1}\left(\left\{x \in \partial\Omega(t_0) : \ V_n(t_0, x) > 0, \ |p|^2 < 1 + \mu_+ \quad \text{for some} \quad p \in D_+u(t_0, x)\right\}\right) > 0.$$

Then there exists $c > 0$ such that the following holds: for all sufficiently small $\delta = t_0 - t_{-1} > 0$ there is a function $u_\delta \in H^1(U)$ with $u_\delta \geq 0$ and $u_\delta = F(t_0)$ on ∂U such that

$$\Omega(u(t_{-1})) \subset \Omega(u_\delta) \subset \Omega(u(t_0)), \ |\Omega(u(t_0)) \setminus \Omega(u_\delta)| \geq c\delta \tag{7.3}$$

and

$$\int_{\Omega(u_\delta)} |\nabla u_\delta|^2 \, dx - \int_{\Omega(t_0)} |\nabla u(t_0)|^2 \, dx \leq (1 + \mu_+ - c)|\Omega(u(t_0)) \setminus \Omega(u_\delta)|. \tag{7.4}$$

Remark 7.5. The constant $c > 0$ in above Lemma is not very easily quantified, it depends delicately on the hypothesis that $\mathcal{H}^{d-1}(\Gamma^+(u, t_0)) > 0$ (not just on the measure itself, but on the consequences of being positive measure).

We defer the proof of Lemma 7.4 to the end of the section. It is worth emphasizing that this part is the technical heart of the proof of Theorem 7.1.

Now we give the proof of the weak solution property Theorem 7.1 via an energy argument using Lemma 7.4. Proposition 5.16 also plays a central role allowing us to reduce to the case of left continuous times.

Proof of Theorem 7.1. Fix $t_0 \in (0,T]$, we first assume that $P(u(t))$ is left continuous at t_0. Jump times will be considered at the end, building on the result at left continuous times and using the upper semi-continuity of u as well as the other regularity properties of energy solutions from Proposition 5.16.

Step 1. Assume for now that

$$|P(u(t_0)) - P(u(t))| \leq \sigma(t_0 - t) \quad \text{for} \quad t \leq t_0$$

with some modulus of continuity σ. Suppose that

$$\mathcal{H}^{d-1}\left(\Gamma^+(u,t)\right) > 0.$$

Then Lemma 7.4 yields u_δ, t_{-1}, t_0 for small $\delta > 0$. We define

$$v(x) := \frac{F(t_{-1})}{F(t_0)} u(t_0, x) \quad \text{and} \quad v_\delta(x) := \frac{F(t_{-1})}{F(t_0)} u_\delta(x).$$

From the energy dissipation inequality (1.6) evaluated on $[t_{-1}, t_0]$, we derive

$$\begin{aligned} \mathrm{Diss}(\Omega(t_{-1}), \Omega(t_0)) &\leq \mathcal{J}(u(t_{-1})) - \mathcal{J}(u(t_0)) + \int_{t_{-1}}^{t_0} 2\dot{F}(t)P(t)\, dt \\ &= [\mathcal{J}(u(t_{-1})) - \mathcal{J}(v_\delta(t_0))] + [\mathcal{J}(v_\delta(t_0)) - \mathcal{J}(v(t_0))] \\ &\quad + \left[\mathcal{J}(v(t_0)) - \mathcal{J}(u(t_0)) + \int_{t_{-1}}^{t_0} 2\dot{F}(t)P(t)\, dt\right] \\ &= A + B + C. \end{aligned} \tag{7.5}$$

We next explain, individually, how to bound A, B, C to yield a contradiction.

Since $v_\delta = F(t_{-1})$ on ∂U, we can apply the stability property Definition 1.1(2) at time t_{-1} to obtain

$$A = \mathcal{J}(u(t_{-1})) - \mathcal{J}(v_\delta) \leq \mathrm{Diss}(\Omega(t_{-1}), \Omega_\delta(t_0)).$$

Next the energy inequality (7.4) from Lemma 7.4 yields

$$B := \mathcal{J}(v_\delta(t_0)) - \mathcal{J}(v(t_0)) \leq \left(\frac{F(t_{-1})^2}{F(t_0)^2}(1 + \mu_+ - c) - 1\right) |\Omega(t_0) \setminus \Omega_\delta(t_0)|.$$

Recall that $c > 0$ is independent of $\delta > 0$, so from the Lipschitz continuity of $\log F(t)$ we conclude that

$$B \leq (\mu_+ - c + o_\delta(1))|\Omega(t_0) \setminus \Omega_\delta(t_0)|.$$

Lastly we address C. We have

$$\begin{aligned}\mathcal{J}(u(t_0)) - \mathcal{J}(v(t_0)) &= \int_U (1 - \frac{F(t_{-1})}{F(t_0)})|Du|^2(t_0)dx \\ &= \frac{F^2(t_0) - F^2(t_{-1})}{F(t_0)} P(t_0) \\ &\geq \int_{t_{-1}}^{t_0} 2\dot{F}(t)P(t)dt - 2\|\dot{F}\|_{L^\infty}\sigma(t_0 - t_{-1})|t_0 - t_{-1}|, \\ &= \int_{t_{-1}}^{t_0} 2\dot{F}(t)P(t)dt - o_\delta(1)\delta. \qquad (7.6)\end{aligned}$$

This is the key place where we use the left continuity hypothesis on P.

Now, combining the previous three inequalities into (7.5), we have obtained

$$\mathrm{Diss}(\Omega(t_{-1}), \Omega(t_0)) \leq \mathrm{Diss}(\Omega(t_{-1}), \Omega_\delta(t_0)) + (\mu_+ - c + o_\delta(1))|\Omega(t_0) \setminus \Omega_\delta(t_0)| + o_\delta(1)\delta$$

On the other hand, since $\Omega(u(t_{-1})) \subset \Omega_\delta(t_0) \subset \Omega(t_0)$ from Lemma 7.4, we have

$$\begin{aligned}\mathrm{Diss}(\Omega(t_{-1}), \Omega(t_0)) - \mathrm{Diss}(\Omega(t_{-1}), \Omega_\delta(t_0)) &= \mu_+|\Omega(t_0) \setminus \Omega_\delta(t_{-1})| - \mu_+|\Omega_\delta(t_0) \setminus \Omega(t_{-1})| \\ &= \mu_+|\Omega(t_0) \setminus \Omega_\delta(t_0)|.\end{aligned}$$

Putting the above estimates together, it follows that

$$\mu_+|\Omega(t_0) \setminus \Omega_\delta(t_0)| \leq (\mu_+ - c + o_\delta(1))|\Omega(t_0) \setminus \Omega_\delta(t_0)| + o_\delta(1)\delta.$$

Since Lemma 7.4 crucially guarantees that $|\Omega(t_0) \setminus \Omega_\delta(t_0)| \geq c\delta$, we can divide through by $|\Omega(t_0) \setminus \Omega_\delta(t_0)|$ and find

$$\mu_+ \leq \mu_+ - c + o_\delta(1)$$

Sending $\delta = (t_0 - t_{-1})$ to zero yields $0 \leq -c$, which is a contradiction.

Step 2. (Jump times) Suppose that $P(u(t))$ is not left continuous at t_0. By Proposition 5.16 (i) u has left and right limits at time t_0, $u_\ell(t_0)$ and $u_r(t_0)$. Furthermore, by Proposition 5.16 part (iii) and since u is upper semicontinuous we must have $u(t_0) = u_\ell(t_0) \vee u_r(t_0)$.

By Proposition 5.16 part (iv)

$$\tilde{u}(t) := \begin{cases} u(t) & t < t_0 \\ u_\ell(t_0) & t = t_0. \end{cases}$$

is itself an energy solution on $[0, t_0]$ and $\tilde{u}(t)$ and $P(\tilde{u}(t))$ are left continuous at t_0. The previous arguments give

$$\mathcal{H}^{d-1}(\Gamma^+(\tilde{u}, t_0)) = 0.$$

We aim to show $\Gamma^+(u, t_0) \subset \Gamma^+(\tilde{u}, t_0)$ which will complete the proof showing $\mathcal{H}^{d-1}(\Gamma^+(u, t_0)) = 0$. By Corollary 7.3

$$\Gamma^+(u, t_0) \setminus \partial\Omega(u_\ell(t_0)) = \emptyset.$$

On the other hand since u touches $\tilde{u}$ from above at each $x_0 \in \partial\Omega_\ell(t_0) \cap \partial\Omega(t_0)$

$$V_n^+(t_0, x_0) \geq \tilde{V}_n^+(t_0, x_0) \quad \text{and} \quad D_+u(t_0, x_0) \subset D_+\tilde{u}(t_0, x_0)$$

Thus

$$\Gamma^+(u, t_0) \subset \Gamma^+(\tilde{u}, t_0),$$

which concludes the proof. $\square$

Remark 7.6. The regularity of the pressure $P(t)$ seems to play an important role. In (7.6) the regularity of $P(t)$ dictates the size of a positive error term which needs to be balanced by the positive $\mathcal{H}^{d-1}$ measure of the set where the viscosity solution condition fails.

Before continuing to the proof of the barrier construction Lemma 7.4, it is useful to separate out one further measure-theoretic Lemma which says that if the subsolution condition fails on a set of positive surface measure, then it fails so quantitatively.

Let us first define quantified versions of the velocity and sub/superdifferentials. For given $r_0, c, \sigma > 0$ we define

$$D_+^{\sigma,r_0}u(x_0) := \left\{p \in \mathbb{R}^d : u(x) \le p \cdot (x - x_0) + \sigma|x - x_0| \ \text{ in } \ B_{r_0}(x_0) \cap \overline{\{u > 0\}}\right\},$$

and

$$D_-^{\sigma,r_0}u(x_0) := \left\{p \in \mathbb{R}^d : u(x) \ge p \cdot (x - x_0) - \sigma|x - x_0| \ \text{ in } \ B_{r_0}(x_0) \cap \overline{\{u > 0\}}\right\}.$$

Lastly we define a quantified set where the dynamic slope condition strictly fails:

$$\Gamma^+_{\eta,\sigma,r_0}(u,t) := \left\{x \in \partial\Omega(u(t)) : \begin{array}{c} V_n^{r_0}(t,x) \ge \eta \ \text{ and } \ \exists p \in D_+^{\sigma,r_0}u(t,x) \\ \text{s.t. } |p|^2 - (1 + \mu_+) < -\eta \end{array} \right\}. \tag{7.7}$$

The subdifferential version is defined similarly. In the above definition we are keeping more parameters than minimally necessary in order to clarify certain delicate points in the proof.

Lemma 7.7. *The family of sets $\Gamma^+_{\eta,r_0,\sigma}(u,t)$ is monotone decreasing with respect to η and r_0 and monotone increasing with respect to σ. Furthermore*

$$\Gamma^+(u,t) = \bigcup_{\eta>0} \bigcap_{\sigma>0} \bigcup_{r_0>0} \Gamma^+_{\eta,\sigma,r_0}(u,t)$$

and thus, by monotone convergence theorem, if $\mathcal{H}^{d-1}(\Gamma^+(u,t)) > 0$ then there is $\eta > 0$ so that for all $\sigma > 0$ there is $r_0(\sigma) > 0$ such that $\mathcal{H}^{d-1}(\Gamma^+_{\eta,\sigma,r_0}(u,t)) > 0$.

Proof. The monotonicities follow from the set definition. Let us check the set formula. Let $x \in \Gamma^+(u,t)$, which means

$$V_n(t,x) > 0 \ \text{ and there is } p \in D_+u(t,x) \text{ with } \ |p|^2 < 1 + \mu_+.$$

By definition of D_+u

$$u(t,x) \le p \cdot x + o(|x - x_0|) \ \text{ for } \ x \in \overline{\Omega(u(t))}$$

and so for every $\sigma > 0$ there is $r_1(\sigma) > 0$ sufficiently small so that

$$u(t,x) \le p \cdot x + \sigma|x - x_0| \ \text{ for } \ x \in \overline{\Omega(u(t))} \cap B_{r_1}(x)$$

Namely $p \in D_+^{\sigma,r_1}u(t,x)$. By the definition of V_n

$$V_n^{r_2}(t,x) \ge \eta_2 > 0 \ \text{ for some } \ r_2, \eta_2 > 0.$$

Call $\eta_1 := (1 + \mu_+) - |p|^2 > 0$. Taking $r_0 = \min\{r_1, r_2\}$ and $\eta = \min\{\eta_1, \eta_2\}$ we find that $x \in \Gamma^+_{\eta,\sigma,r_0}(u,t)$. ∎

Proof of Lemma 7.4. By Lemma 7.7 and the hypothesis of the Lemma there is $\eta > 0$ such that for all $\sigma > 0$ there is positive $r_0(\sigma)$ with

$$\mathcal{H}^{d-1}(\Gamma^+_{\eta,\sigma,r_0}(u,t_0)) \geq \frac{1}{2}\mathcal{H}^{d-1}(\Gamma^+(u,t_0)) > 0. \tag{7.8}$$

We will take $\sigma = c(d)\eta$, where the dimensional constant $c(d)$ will be determined in the course of the proof (under (7.15)). This choice of σ also fixes $r_0 = r_0(d,\eta) > 0$, we will usually drop the dependence on the dimension when it is not important and write (for example) $r_0(\eta)$. Let us consider $\delta \in (0, r_0(\eta)]$ and $t_{-1} := t_0 - \delta$ for the rest of the proof. By monotonicity of the sets $\Gamma^+_{\eta,\sigma,r_0}(u,t_0)$ in η, we may assume that $\eta < \frac{1}{2}$.

We outline the proof, which is divided into three steps. In the first step we use the speed and slope condition at a single point to construct a strict supersolution replacement in a local neighborhood which is below $u(t_0)$ by $O(\delta^d)$ in measure but still above $u(t_{-1})$. The construction is fairly standard, using the first order asymptotic information from the quantified superdifferential, sliding the linearization inwards while also bending up in the tangential directions and bending down in the normal direction in order to create a local superharmonic perturbation. In the second step we perform a typical Vitali-type covering argument. In the final step we perform the local supersolution replacement in each disjoint ball from the covering to create the perturbation u_δ, which is now $O(\delta)$ below $u(t_0)$ because we have shifted inwards by $O(\delta)$ on a set of positive surface measure.

Step 1. (Construction of a barrier based on single reference point) In order to make certain parameter choices more clear in the proof below we will adopt the notation $\eta_t = \eta_x = \eta$ and use η_t when the lower bound of the velocity is being used, and η_x when the lower bound of the slope condition is being used. Let $x_0 \in \Gamma^+_{\eta,\sigma,r_0}(u,t_0)$. For simplicity we may assume that $x_0 = 0$ and $p_0 = \alpha e_d$ for some $\alpha \geq 0$, so that

$$u(t_0,x) \leq \alpha x_d + \sigma|x| \quad \text{in} \quad B_{r_0}(0) \cap \overline{\Omega(t_0)} \tag{7.9}$$

and, since we can always increase α if necessary, we may assume that

$$\alpha = (1+\mu_+)^{1/2} - \eta_x.$$

Note that, since u is nonnegative, (7.9) yields that $\alpha(x_d)_- \leq \sigma r$ on $B_r(0) \cap \overline{\Omega(t_0)}$, namely

$$B_r(0) \cap \overline{\Omega(t_0)} \subset \{x_d \geq -\frac{\sigma}{\alpha} r\} \quad \text{for all} \quad r \leq r_0. \tag{7.10}$$

Furthermore, again from the definition of $\Gamma^+_{\eta,\sigma,r_0}(u,t_0)$,

$$B_{\eta_t(t_0-t)}(0) \subset \mathbb{R}^d \setminus \Omega(t) \quad \text{for} \quad t \geq t_0 - r_0. \tag{7.11}$$

Recall that we assumed that $\delta = t_0 - t_{-1}$ is smaller than $r_0(\eta) > 0$.

Our perturbation will be performed in e_d-aligned cylinders which are anisotropic, slightly wider in the tangential directions than the normal direction. Denoting $x' = (x_1, \dots, x_{d-1})$ and B'_r the ball of radius r in $\mathbb{R}^{d-1}$,

$$\mathrm{Cyl}_r = B'_r \times (-c_d r, c_d r).$$

The dimensional constant $c_d := \frac{1}{\sqrt{2(d-1)}}$ is chosen so that on the side boundary of the cylinder $\partial_{side}\mathrm{Cyl}_r := \partial B_r \times (-c_d r, c_d r)$

$$|x'|^2 - (d-1)x_d^2 \geq r^2 - (d-1)c_d^2 r^2 \geq \frac{1}{2}r^2 \quad \text{on} \quad x \in \partial_{side}\mathrm{Cyl}_r. \tag{7.12}$$

We also call

$$\partial_{top}\mathrm{Cyl}_r := B_r \times \{x_d = c_d r\} \quad \text{and} \quad \partial_{bot}\mathrm{Cyl}_r := B_r \times \{x_d = -c_d r\}.$$

Obviously $\mathrm{Cyl}_r \subset B_{r_0}$ for $r \leq r_0/\sqrt{2}$.

We aim to make a localized perturbation in $\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}$. Note that this cylinder is chosen since, by (7.11),

$$\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta} \subset B_{\eta_t\delta} \subset \mathbb{R}^d \setminus \Omega(t_{-1}). \tag{7.13}$$

Since we assume $\delta = t_0 - t_{-1} \leq r_0$ and $\eta < 1$, so $\eta_t\delta \leq r_0$ as well.

We shift the plane αx_d inwards by $O(\delta)$ in the normal direction e_d while also bending upwards in the tangential directions so that the barrier is below $u(t_0)$ but above $u(t_{-1})$.

More precisely, define the barrier

$$\psi_\delta(x) := (\alpha + \tfrac{\eta_x}{4})(x_d - a_1\eta_t\delta) + a_2(\eta_t\delta)^{-1}(|x'|^2 - (d-1)x_d^2). \tag{7.14}$$

We will see that if we choose $a_i := c(d, \mu_+)\eta_x, i = 1, 2$ then we can guarantee the following important properties of ψ_δ:

(i) ψ_δ is harmonic;

(ii) $\psi_\delta(x) > u(t_0, x)$ on $\partial\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta} \cap \overline{\Omega(t_0)}$;

(iii) $|\nabla\psi_\delta(x)|^2 \leq 1 + \mu_+ - \frac{1}{2}\eta_x$ in $\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}$;

(iv) There is $c > 0$ depending on d and μ_+ such that

$$B_{c\eta_x\eta_t\delta}(0) \subset \{\psi_\delta \leq 0\}.$$

Property (i). This is straightforward, but we do emphasize that the superharmonicity is important and needed later to deal with the energy computation in Step 3 below.

Property (ii). We check the bottom, sides and top of the cylinder separately. On the top boundary $x \in \partial_{top}\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}$ we can use the extra room on the slope:

$$\begin{aligned}\psi_\delta(x) &\geq \alpha x_d + c(d)\eta_x\eta_t\delta - C(d)(a_1 + a_2)\eta_t\delta \\ &\geq \alpha x_d + c(d)[\eta_x - C(d)(a_1 + a_2)]\eta_t\delta.\end{aligned}$$

Thus if we take

$$a_1 + a_2 \leq c(d)\eta_x \tag{7.15}$$

then, again on $x \in \partial_{top}\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}$,

$$\begin{aligned}\psi_\delta(x) &\geq \alpha x_d + c(d)\eta_x\eta_t\delta \\ &\geq \alpha x_d + c(d)\eta_x|x| \geq \alpha x_d + \sigma|x| \geq u(t_0, x).\end{aligned}$$

where this computation fixes the choice of the dimensional constant in $\sigma = c(d)\eta_x$.

On the sides $\partial_{side}\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta} \cap \overline{\Omega(t_0)}$ we can use the upwards bending in the tangential directions. Using (7.12), we have.

$$\begin{aligned}\psi_\delta(x) &\geq \alpha x_d - \tfrac{1}{4}\eta_x(x_d)_- - (1+\mu_+)^{1/2}a_1\eta_t\delta + \frac{1}{2}a_2\eta_t\delta \\ &= \alpha x_d + \sigma|x| + \underbrace{\left[\tfrac{1}{2}a_2\eta_t\delta - \sigma|x| - \tfrac{1}{4}\eta_x(x_d)_- - (1+\mu_+)^{1/2}a_1\eta_t\delta\right]}_{:=A}\end{aligned}$$

(7.10) yields that

$$A \geq c\left[a_2 - C(d)\sigma - C(\mu_+, d)a_1\right]\eta_t\delta \text{ on } x \in \partial_{side}\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta} \cap \overline{\Omega(t_0)}.$$

We want this to be non-negative, so along with condition (7.15) we need to choose σ, a_1 and a_2 so that

$$a_1 + a_2 \leq c(d)\eta_x \quad \text{and} \quad C(d)\sigma + C(\mu_+, d)a_1 \leq a_2.$$

To satisfy both inequalities, we choose $\sigma = c(d)\eta_x$, $a_1 = c(d,\mu_+)\eta_x$ and $a_2 = c(d)\eta_x$. Applying these choices of parameters, we find

$$\psi_\delta(x) \geq \alpha x_d + \sigma|x| \geq u(t_0, x) \quad \text{on} \quad \partial_{side}\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta} \cap \overline{\Omega(t_0)}.$$

Finally, the bottom boundary $\partial_{bot}\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}$ is compactly contained in the zero level set of $u(t_0)$. Namely the set $(x_d)_- = c_d\eta_t\delta$ does not intersect $\overline{\Omega(t_0)}$ by (7.10) and fixing the choice of dimensional constant in the original specification $\sigma = c(d)\eta$.

Property (iii). For this we just compute the gradient and bound with triangle inequality

$$|\nabla\psi_\delta - (\alpha + \tfrac{1}{4}\eta_x)e_d| \leq C(d)a_2 \quad \text{ in } \mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}.$$

Choosing the dimensional constant in $a_2 = c(d)\eta_x$ smaller if necessary,

$$|\nabla\psi_\delta| \leq \alpha + \tfrac{1}{2}\eta_x \leq (1+\mu_+)^{1/2} - \tfrac{1}{2}\eta_x \quad \text{ in } \mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}.$$

Property (iv). Note that ψ_δ is monotone in $\mathrm{Cyl}_{\frac{1}{\sqrt{2}}\eta_t\delta}$ on the cone of directions, namely for the directions $e \in S^{d-1}$ such that $e \cdot e_d \geq \frac{1}{2} \geq C(d)a_2$. Since $\psi_\delta(a_1\eta_t\delta e_d) = 0$ we find that

$$B_{2a_1\eta_t\delta e_d}(0) \subset \{\psi_\delta \leq 0\}.$$

Step 2. (Covering set up) Since Hausdorff measure is inner regular, there is a compact subset $K \subset \Gamma^+_{\eta,\sigma,r_0}(u,t_0)$ with

$$\mathcal{H}^{d-1}(K) \geq \frac{1}{2}\mathcal{H}^{d-1}(\Gamma^+_{\eta,\sigma,r_0}(u,t_0)) \geq \frac{1}{4}\mathcal{H}^{d-1}(\Gamma^+(u,t_0)).$$

Since K is compact, there is $r_1 > 0$ such that

$$\inf\{\sum_{j=1}^{\infty}\rho_j^{d-1} : K \subset \bigcup_{j=1}^{\infty} B_{\rho_j}(x_j),\ \rho_j \leq r_1\} \geq \frac{1}{2}\mathcal{H}^{d-1}(K). \tag{7.16}$$

Let $3\eta_t\delta \leq r_1$. By the Vitali covering lemma there is a finite collection of disjoint balls B_j of radius $\eta_t\delta$ centered at points $x_j \in K \subset \partial\Omega(t_0)$ with

$$K \subset \bigcup_j 3B_j.$$

Call $\tilde{B}_j = B_{c\eta_x\eta_t\delta}(x_j)$ with constant $1 > c(d,\mu_+) > 0$ from property (iv) above. Since $\tilde{B}_j \subset B_j$ they are a disjoint collection as well. By (7.16)

$$\sum_j 3^{d-1}\delta^{d-1} \geq \frac{1}{2}\mathcal{H}^{d-1}(K).$$

By measure non-degeneracy of the free boundary Lemma 5.5

$$|\Omega(t_0) \cap \tilde{B}_j| \geq c\eta^{-2d}\delta^d,$$

where the constant c depends on d and μ_+.

Since $\tilde{B}_j's$ are disjoint, we obtain the following by absorbing universal constants into c:

$$|\Omega(t_0) \cap \bigcup_j \tilde{B}_j| = \sum_j |\Omega(t_0) \cap \tilde{B}_j| \geq \sum_j c\eta^{-2d}\delta^d \geq c\eta^{-2d}\mathcal{H}^{d-1}(K)\delta,$$

or,

$$|\Omega(t_0) \cap \bigcup_j \tilde{B}_j| \geq c\eta^{-2d}\mathcal{H}^{d-1}(\Gamma^+(u,t_0))\delta. \tag{7.17}$$

Step 3. (Construction of the comparison functions) Let $\delta \leq \min\{(3\eta_t)^{-1}r_1, r_0\}$ so that both Step 1 and Step 2 apply, recall r_1 was fixed by (7.16) and $r_0(\eta)$ was fixed at the beginning of the proof below (7.8). Now for each $x_j \in \Gamma^+(u,t_0)$ from the covering constructed in Step 2, call $\psi_{x_j,\delta}$ to be the single-point barrier function constructed in Step 1. The $\psi_{x_j,\delta}$ are defined in B_j, extend them to be equal to $+\infty$ outside of their respective B_j.

Consider the comparison function

$$u_\delta(x) = u(t_0,x) \wedge \bigwedge_j \psi_{x_j,\delta}(x) \ \text{ defined for } \ x \in \Omega(t_0).$$

By properties (i) and (ii), and by the disjointness of the B_j, we can conclude that u_δ is Lipschitz continuous and superharmonic in $\Omega(t_0)$.

Now by property (iv) (first containment below), property (ii) (second containment below), and (7.13) (third containment below), applied for each j

$$\bigcup_j \tilde{B}_j \cap \Omega(u(t_0)) \subset \Omega(u(t_0)) \setminus \Omega(u_\delta) \subset \bigcup_j B_j \subset \mathbb{R}^d \setminus \Omega(u(t_{-1})). \tag{7.18}$$

Thus we conclude that

$$\Omega(u(t_{-1})) \subset \Omega(u_\delta) \subset \Omega(u(t_0))$$

and also, by (7.17),

$$|\Omega(u(t_0)) \setminus \Omega(u_\delta)| \geq \sum_j |\tilde{B}_j \cap \Omega(u(t_0))| \geq c\eta^{-2d}\mathcal{H}^{d-1}(\Gamma^+(u,t_0))\delta.$$

The previous two displayed equations establish (7.3), the first claim of the Lemma.

To conclude we still need to establish the inequality on the difference of the Dirichlet energy (7.4). Recall that u_δ is superharmonic and Lipschitz continuous in $\Omega(t_0)$, so we can apply Lemma A.2 with $v_0 = u_\delta$ and $v_1 = u(t_0)$. This results in

$$\int_{\Omega(u_\delta)} |\nabla u_\delta|^2 \, dx - \int_{\Omega(t_0)} |\nabla u(t_0)|^2 \, dx \leq \int_{\Omega(t_0)\setminus\Omega(u_\delta)} |Du_\delta|^2 \, dx$$
$$\leq \int_{\Omega(t_0)\setminus\Omega(u_\delta)} (1 + \mu_+ - \tfrac{1}{2}\eta_x) \, dx$$

where we applied property (iii) for the second inequality. This proves (7.4) and finishes the proof of the Lemma.

□

Appendix A. Technical results

A.1. **Non-degeneracy.** Non-degeneracy of Bernoulli problem solutions is much more delicate than Lipschitz continuity, it typically needs a stronger condition than just the viscosity solution property. Here we give several cases where it is known.

Lemma A.1. *Let $u \in C(B_2)$ satisfy one of the following:*

(i) Suppose that $\partial\{u > 0\}$ is an L-Lipschitz graph in B_2 and u solves

$$\Delta u = 0 \quad \text{in} \quad \{u > 0\} \cap B_2, \quad \text{and} \quad |\nabla u| \geq 1 \text{ on } \partial\{u > 0\} \cap B_2.$$

(ii) Suppose that u is an inward minimizer of $\mathcal{J}$ in B_2.

(iii) Suppose that u is a minimal supersolution in B_2

$$\Delta u = 0 \quad \text{in} \quad \{u > 0\} \cap B_2, \quad \text{and} \quad |\nabla u| \leq 1 \text{ on } \partial\{u > 0\} \cap B_2.$$

(iv) Suppose $d = 2$ and u is a maximal subsolution in B_2 of

$$\Delta u = 0 \quad \text{in} \quad \{u > 0\} \cap B_2, \quad \text{and} \quad |\nabla u| \geq 1 \text{ on } \partial\{u > 0\} \cap B_2.$$

Then there is c_0 depending on d (and on L if in the first case) such that

$$\sup_{B_r(x)} u(x) \geq c_0 r \quad \text{for all } x \in \partial\{u > 0\} \cap B_1, \; B_r(x) \subset B_2.$$

Proof. For part (i) see [15, Lemma 6.1], for part (ii) see [36, Lemma 4.4], for part (iii) see [6, Lemma 6.9], and for (iv) see [33]. □

A.2. **Energy difference quotient formulae.** We give and prove several formulas for the difference of the Dirichlet energy under varying positivity set. We have not seen these particular formulas before in the literature, but they provide a very convenient way to go from energy minimization to viscosity solution properties.

Lemma A.2. *Suppose that $v_0, v_1 \in H^1(U)$ with $v_0 \leq v_1$ and $v_0 = v_1 \geq 0$ on ∂U. Call $\Omega_j = \{v_j > 0\} \cap U$.*

- *If v_1 is subharmonic in Ω_1 then*

$$\int_{\Omega_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx \geq \int_{\Omega_1\setminus\Omega_0} |\nabla v_1|^2 \, dx. \tag{A.1}$$

- *If v_0 is superharmonic in Ω_1 then*

$$\int_{\Omega_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx \leq \int_{\Omega_1\setminus\Omega_0} |\nabla v_0|^2 \, dx \tag{A.2}$$

Proof. For (A.1) we start with

$$\int_{\Omega_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx = \int_{\Omega_0} |\nabla v_0|^2 - |\nabla v_1|^2 \, dx + \int_{\Omega_1 \setminus \Omega_0} -|\nabla v_1|^2 \, dx$$

Then we compute the first term on the right

$$\begin{aligned}
\int_{\Omega_0} |\nabla v_0|^2 - |\nabla v_1|^2 \, dx &= \int_{\Omega_0} |\nabla v_1 + \nabla(v_0 - v_1)|^2 - |\nabla v_1|^2 \, dx \\
&= \int_{\Omega_0} 2\nabla v_1 \cdot \nabla(v_0 - v_1) + |\nabla(v_0 - v_1)|^2 \, dx \\
&\geq \int_{\Omega_0} 2\nabla v_1 \cdot \nabla(v_0 - v_1) \, dx \\
&= \int_{\Omega_1} 2\nabla v_1 \cdot \nabla((v_0)_+ - v_1) \, dx + \int_{\Omega_1 \setminus \Omega_0} 2|\nabla v_1|^2 \, dx \\
&\geq \int_{\Omega_1 \setminus \Omega_0} 2|\nabla v_1|^2 \, dx.
\end{aligned}$$

The last inequality in the sequence above is is due to v_1 being H^1 subharmonic in Ω_1. Note that $v_1 - (v_0)_+$ is a valid test function for the subharmonicity condition since it is non-negative in Ω_1 and in $H^1_0(\Omega_1)$.

For (A.2) we start with

$$\int_{\Omega_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx = \int_{\Omega_1} |\nabla v_0|^2 - |\nabla v_1|^2 \, dx + \int_{\Omega_1 \setminus \Omega_0} -|\nabla v_0|^2 \, dx$$

Then we compute the first term on the right

$$\begin{aligned}
\int_{\Omega_1} |\nabla v_0|^2 - |\nabla v_1|^2 \, dx &= \int_{\Omega_1} |\nabla v_0|^2 - |\nabla v_0 + \nabla(v_1 - v_0)|^2 \, dx \\
&= \int_{\Omega_1} -2\nabla v_0 \cdot \nabla(v_1 - v_0) - |\nabla(v_1 - v_0)|^2 \, dx \\
&\leq \int_{\Omega_1} -2\nabla v_0 \cdot \nabla(v_1 - (v_0)_+) \, dx + \int_{\Omega_1 \setminus \Omega_0} 2|\nabla v_0|^2 \, dx \\
&\leq \int_{\Omega_1 \setminus \Omega_0} 2|\nabla v_0|^2 \, dx.
\end{aligned}$$

The last inequality in the sequence above is is due to v_0 being H^1 superharmonic in Ω_1. Note that $v_1 - (v_0)_+$ is a valid test function for the superharmonicity condition since it is non-negative in Ω_1 and in $H^1_0(\Omega_1)$.

In the smooth case one can also check the related identities

$$\begin{aligned}
&\int_{\Omega_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx \\
&\quad = \int_{\Omega_1} 2\Delta v_1 (v_1 - (v_0)_+) \, dx + \int_{\Omega_0} |\nabla(v_1 - v_0)|^2 \, dx + \int_{\Omega_1 \setminus \Omega_0} |\nabla v_1|^2 \, dx.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_0} |\nabla v_0|^2 \, dx - \int_{\Omega_1} |\nabla v_1|^2 \, dx = \\
&\quad \int_{\Omega_1} 2(v_1 - (v_0)_+)\Delta v_0 \, dx - \int_{\Omega_1} |\nabla(v_1 - v_0)|^2 \, dx + \int_{\Omega_1 \setminus \Omega_0} |\nabla v_0|^2 \, dx.
\end{aligned}$$

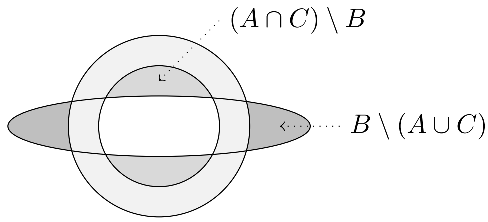

FIGURE 8. Diagram of the sharp triangle inequality for the dissipation distance. Here $A \cap C$ is the inner circle, $A \cup C$ is the outer circle, and B is the ellipse.

□

A.3. **Sharp triangle inequality of the Dissipation distance.** Here we give a sharp triangle inequality for the Dissipation distance. The triangle inequality is important for most of the standard theory of rate independent energetic evolutions, although we remark that it does not seem to generalize well to the general anisotropic case. The sharp triangle inequality is used in Proposition 5.16 to show that if an energetic solution jumps multiple times then $u(t)$ must always be in between $u_\ell(t)$ and $u_r(t)$.

Lemma A.3 (Dissipation distance sharp triangle inequality)**.** *Let A, B, and C be arbitrary finite measure regions in $\mathbb{R}^n$. Then*

$$\mathrm{Diss}(A,B) + \mathrm{Diss}(B,C) - \mathrm{Diss}(A,C) = (\mu_- + \mu_+)\Big[|B \setminus (A \cup C)| + |(A \cap C) \setminus B|\Big]$$

Proof. The result is purely set algebraic computations, see Figure 8 for the geometric idea. We compute

$$\begin{aligned}
&\mathrm{Diss}(A,B) + \mathrm{Diss}(B,C) - \mathrm{Diss}(A,C) \\
&= \int \mu_+ \mathbf{1}_{B\setminus A} + \mu_- \mathbf{1}_{A\setminus B} + \mu_+ \mathbf{1}_{C\setminus B} + \mu_- \mathbf{1}_{B\setminus C} - \mu_+ \mathbf{1}_{C\setminus A} - \mu_- \mathbf{1}_{C\setminus A}\, dx \\
&= \int \mu_+[\mathbf{1}_{B\setminus A} + \mathbf{1}_{C\setminus B} - \mathbf{1}_{C\setminus A}] + \mu_-[\mathbf{1}_{A\setminus B} + \mathbf{1}_{B\setminus C} - \mathbf{1}_{A\setminus C}]\, dx. \qquad \text{(A.3)}
\end{aligned}$$

Now we just compute by force the first term in brackets on the previous line

$$\begin{aligned}
\mathbf{1}_{B\setminus A} + \mathbf{1}_{C\setminus B} - \mathbf{1}_{C\setminus A} &= \mathbf{1}_B - \mathbf{1}_{A\cap B} + \mathbf{1}_C - \mathbf{1}_{B\cap C} - [\mathbf{1}_C - \mathbf{1}_{A\cap C}] \\
&= \mathbf{1}_B + \mathbf{1}_{A\cap C} - \mathbf{1}_{B\cap C} - \mathbf{1}_{B\cap A} \\
&= \mathbf{1}_B + \mathbf{1}_{A\cap C}\mathbf{1}_B - \mathbf{1}_{B\cap C} - \mathbf{1}_{B\cap A} + \mathbf{1}_{A\cap C}(1 - \mathbf{1}_B) \\
&= \mathbf{1}_B(1 + \mathbf{1}_{A\cap C} - \mathbf{1}_C - \mathbf{1}_A) + \mathbf{1}_{(A\cap C)\setminus B} \\
&= \mathbf{1}_B(1 - \mathbf{1}_{A\cup C}) + \mathbf{1}_{(A\cap C)\setminus B} \\
&= \mathbf{1}_{B\setminus(A\cup C)} + \mathbf{1}_{(A\cap C)\setminus B}
\end{aligned}$$

the second bracketed term in (A.3) works out quite similarly and is identical after the second step above giving the same result.

Plugging these identities back into (A.3) gives the result. □

A.4. **Proof of Lemma 4.7.** We just do the subsolution equivalence, the supersolution one is similar. Suppose u is a subsolution in the superdifferential sense Definition 4.5. Suppose φ is a smooth test function with φ_+ touching u from above at $x_0 \in \partial\Omega(u) \cap U$. Then

$$u(x) \leq [\nabla\varphi(x_0) \cdot (x - x_0)]_+ + o(|x - x_0|) \quad \text{so} \quad \nabla\varphi(x_0) \in D_+u(x_0)$$

and so

$$|\nabla\varphi(x_0)|^2 \geq Q.$$

Next suppose that u is a subsolution in the superdifferential sense Definition 4.6, and $p \in D_+u(x_0)$ for some $x_0 \in \partial\Omega(u) \cap U$. Without loss assume that $p = \alpha e_d$. Let $\delta, r > 0$ and define

$$\varphi(x) := (1 + \delta)\alpha(x - x_0) \cdot e_d + c(d)\delta r^{-1}(|(x - x_0)'|^2 - dx_d^2).$$

This function is a superharmonic polynomial and using the superdifferential condition $\alpha e_d \in D_+u(x_0)$ one can show that for all $\delta > 0$ there is $r > 0$ so that $u < \varphi$ on $\partial C \cap \overline{\{u > 0\}}$ where C is a small open cylindrical neighborhood of x_0 with radius $\approx r_0$, see the proof of Lemma 7.4 for a similar computation.

Then, since $\varphi(x_0) = u(x_0)$ we can shift φ vertically $\varphi(x) + s$ until $(\varphi(x) + s)_+$ touches u from above at some $x_1 \in C$. Since φ is strictly superharmonic the touching must be on $x_1 \in \partial\Omega(u) \cap C$. Thus finally the viscosity solution condition implies that

$$Q \leq |\nabla\varphi(x_1)|^2 \leq \alpha + o_\delta(1).$$

Sending $\delta \to 0$ finishes the argument. □

A.5. **Uniqueness of the Bernoulli problem solution in the complement of a star-shaped set.** In the star-shaped case we have the following uniqueness result for a Bernoulli problem with an obstacle. The result without obstacle for classical solutions in dimension two was proved in [35] based on an existence theorem by Beurling [4] (more easily found reprinted here [5]). Here we give a proof for viscosity solutions of the obstacle problem for convenience.

Theorem A.4. *Suppose that U is a domain such that that $K = \mathbb{R}^d \setminus U$ is strongly star-shaped with respect to a neighborhood of the origin, and that O is a strongly star-shaped set with respect to the same neighborhood. If $u, v \in C(\overline{U})$ have compact support and for some constants $F, q > 0$ are viscosity solutions of a Bernoulli problem with obstacle O from below*

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap U, \\ u > 0 & \text{in } O, \\ |\nabla u| = q & \text{on } (\partial\{u > 0\} \setminus \overline{O}) \cap U, \\ |\nabla u| \leq q & \text{on } \partial\{u > 0\} \cap U, \\ u = F & \text{on } \partial U \end{cases}$$

then $u = v$.

The same result applies for a Bernoulli problem with obstacle O from above.

Proof. By a simple rescaling it is sufficient to consider $F = q = 1$. We show that $\Omega(u) = \Omega(v)$ which is sufficient by the uniqueness of the Laplace equation.

We first check that $\Omega(u) \subset \Omega(v)$. Suppose that $\Omega(u) \setminus \Omega(v) \neq \emptyset$. Consider the largest $\lambda < 1$ such that

$$v^\lambda(x) := v(\lambda x)$$

satisfies

$$\Omega(u) \subset \Omega(v^\lambda) = \lambda^{-1}\Omega(v).$$

Such λ exists since both $\Omega(u)$ and $\Omega(v)$ are open, $\Omega(u)$ is bounded and $\Omega(v)$ contains a neighborhood of the origin. We have clearly $\lambda < 1$, and by the definition of λ there exists $x_0 \in \partial\Omega(u) \cap \partial\Omega(v^\lambda)$. Also note that $x_0 \notin \overline{O}$ since O is strongly star-shaped and $O \cap \partial\Omega(v) = \emptyset$.

If u and v are sufficiently smooth, we have

$$|\nabla v^\lambda|(x_0) = \lambda|\nabla v|(\lambda x_0) \leq \lambda < 1 = |\nabla u|(x_0),$$

a contradiction with the fact that $u \leq v^\lambda$ by the maximum principle for the harmonic function $v^\lambda - u$ in $\Omega(v) \cap \lambda^{-1}U$ since $v^\lambda - u > 0$ on $\lambda^{-1}\partial U \subset U$ by the strong star-shapedness.

To make the argument work without assuming regularity, we consider the sup and inf convolutions

$$v_r(x) = \inf_{\overline{B}_r(x)} v, \qquad u_r(x) = \sup_{\overline{B}_r(x)} u$$

defined on $\overline{U}_r$ where $U_r := \{x : \overline{B}_r(x) \subset U\}$. It is easy to see that v_r is superharmonic on $U_r \cap \Omega(v_r)$, u_r is subharmonic on $U_r \cap \Omega(u_r)$, and $\inf_{\partial U_r} v_r \to 1$ as $r \to 0$ by continuity.

Let us call the $\lambda < 1$ above λ_0. For each $r > 0$ there exists a largest $\lambda_r < 1$ so that $\Omega(u_r) \subset \lambda_r^{-1}\Omega(v_r)$. Furthermore $\lambda_r \to \lambda_0 < 1$ as $r \to 0$. Therefore we can choose $r > 0$ sufficiently small so that $\lambda_r < 1$, $\lambda_r^{-1/2} \inf_{\partial U_r} v_r > 1$ and $\operatorname{dist}(O, \lambda_r^{-1}\partial\Omega(v_r)) > r$ (by strong star-shapedness of O).

We fix such $r > 0$ and choose $x_0 \in \partial\Omega(u_r) \cap \lambda_r^{-1}\partial\Omega(v_r)$. Let us set $w(x) := {\lambda_r}^{-1/2} v_r(\lambda_r x)$. We have that both u_r and w are defined on $\lambda_r^{-1}\overline{U}_r$. We also note that $w - u_r$ is superharmonic on $\lambda_r^{-1}U_r \cap \{u_r > 0\}$ with $w - u_r > 0$ on $\lambda_r^{-1}\partial U_r$. Therefore the slopes of u_r and w in the normal direction at x_0 are ordered.

We have the standard situation of boundaries of u_r and w touching at a point that has both interior and exterior touching balls centered at the free boundaries of u and $v(\lambda_r x)$ respectively. By a standard barrier construction at these centers, we arrive at a contradiction with the fact that u satisfies $|\nabla u| \geq 1$ (since the center is not in $\overline{O}$ by the choice of r) and $v^{\lambda_r} := v(\lambda_r x)$ satisfies ${\lambda_r}^{-1/2}|\nabla v^{\lambda_r}| \leq \lambda_r^{1/2} 1 < 1$. Therefore we conclude that $\Omega(u) \subset \Omega(v)$.

The inclusion $\Omega(v) \subset \Omega(u)$ can be shown by the same argument, swapping the roles of u and v. By the uniqueness of the Laplace equation we have $u = v$.

The proof for the problem with obstacle from above follows an analogous argument. □

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