NOTE: MATH FOR NAVIER-STOKES AND SPH

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In this note I want to show the minimal rigorous mathematical derivation of Navier-Stokes equations and other conservation laws. This approach is based on the flow map and the change of variables formula to minimize to need for arbitrary assumptions or "intuition". The change of variables formula is a well-known result that is used in many places in mathematics. For a complete proof in a very general context see [EG, Section 3.4.3].

1. Description of the fluid motion

Mathematically, we describe the evolution of a fluid as the motion of material points along trajectories in space and time. Let us fix a dimension $n \in \mathbb{N}$ and we will assume that the fluid fills the whole $\mathbb{R}^n$ at all times. We will assume also that each material point moves along a smooth trajectory and that trajectories do not intersect. We will use the position of material points at $t = 0$ as the "tag" of the material point. That is, if $\xi \in \mathbb{R}^n$ is a fixed Euclidean vector, the *material point* ξ refers to the material point that is located at the position ξ at time $t = 0$. Note that material points do not have any physical meaning besides being the "tracers" that follow the *macroscopic, averaged* motion of the fluid. They are definitely not molecules or atoms of the fluid, whose motion is much more chaotic and complicated.

We will denote the position of the material point ξ at time $t \in \mathbb{R}$ by $X^\xi(t) \in \mathbb{R}^n$. We might drop the superscript ξ from the notation if it is clear from the context. Note that by the tagging explained above, $X^\xi(0) = \xi$.

In the following computations, we will assume that the flow is regular in the following way:

(a) The function (map) $\phi(\xi, t) := X^\xi(t)$ as a function $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth (at least twice continuously differentiable).
(b) The map $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined as $\phi_t(\xi) := \phi(\xi, t) = X^\xi(t)$ is invertible for every $t \in \mathbb{R}$.
(c) The determinant of the Jacobian matrix[1] is nonzero,

$$\det D_\xi \phi(\xi, t) \neq 0 \qquad \text{for all } \xi \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.1}$$

The last condition guarantees that the material points do not become "infinitely" dense as we will see later. The second condition asserts that the fluid fills the whole domain without holes and that no two material points occupy the same space. Mathematically, the above conditions will allow us to use the change of variables formula, differentiate and integrate by parts.

We will refer to the function ϕ as the *flow map*. This map completely captures the kinematic evolution of the fluid.

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[1] $D_\xi \phi(\xi, t)$ is the matrix with entries $\frac{\partial \phi_i}{\partial \xi_j}(\xi, t)$

However, we also want to keep track of other physical properties of the fluid like density, velocity, temperature, etc. There are two ways how to do achieve this:

Eulerian description: Keep track of what the value of the property is at a given *fixed* point $x \in \mathbb{R}^n$ in the physical space at a given time $t \in \mathbb{R}$.

Lagrangian description: Keep track of what the value of the property is at the given material point $\xi \in \mathbb{R}^n$ at a given time $t \in \mathbb{R}$.

The fundamental difference is in *where* in the physical space we focus our attention. In the Eulerian description, we fix a point in the physical space. On the other hand, the position of the material point ξ in the Lagrangian description at time t is $X^\xi(t)$, which moves in the physical space with time. These two descriptions are equivalent in that they both allow us to describe the physical system and we will show how to convert between them, but they differ in how difficult it is to answer various questions about the system:

Change in time: How does a property change with time? Since physical properties are *carried around* (transported) by the fluid, it is easier to understand their change in time if we follow a material point. Here the Lagrangian description is convenient.

Change in space: How does the property change in the physical space? What is the value like on the neighboring material points? In this case, this information is provided much more conveniently by the Eulerian description. In the Lagrangian description, we have to find the neighboring material points first. That is possible, but much less convenient.

To distinguish between the descriptions, we will denote the properties in the Eulerian description by lowercase letters and in the Lagrangian description by uppercase letters. For example, $f(x,t)$ is the value of the property f at point $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ in the Eulerian description, and $F^\xi(t)$ is the value of the same property at time $t \in \mathbb{R}$ on the material point $\xi \in \mathbb{R}^n$. We have the following fundamental relationship between these descriptions:

$$F^\xi(t) = f(X^\xi(t), t) \qquad \text{for all } \xi \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.2}$$

In other words, the value of the physical property f on the material point ξ is the same as the value of this physical property at the point in the physical space $X^\xi(t)$ where the material point ξ is located at time t. Since the flow map $\phi_t(\xi) := X^\xi(t)$ is invertible, we can also write

$$f(x,t) = F^{\phi_t^{-1}(x)}(t) \qquad \text{for all } x \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.3}$$

Recall again that ξ in (1.2) represents the tag of the material point, while x is the position in the physical space in (1.3).

Let us introduce the symbols for the important physical properties[2].

	Eulerian	Lagrangian
position	x	$X^\xi(t)$
velocity	$u(x,t)$	$U^\xi(t)$
density	$\rho(x,t)$	$R^\xi(t)$
$\vdots$		

[2]Note that the uppercase ρ is actually P, but I guess that is more confusing than R.

By definition, the velocity of the fluid at a given point is equal to the velocity of the material point located there. In other words,

$$U^\xi(t) = \frac{d}{dt}X^\xi(t) \qquad \text{for all } \xi \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.4}$$

Thus by (1.2) we have

$$u(X^\xi(t), t) = \frac{d}{dt}X^\xi(t) \qquad \text{for all } \xi \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.5}$$

By differentiating the value of the property in the Lagrangian description with respect to time, $\frac{d}{dt}F^\xi(t)$, we get the rate of change of the property along a material point ξ. This is called also the *material derivative* or the *substantial derivative*. This the rate of change that a given material point observes.

By (1.2), the chain rule and (1.5) we get immediately an expression in terms of the derivatives of the property in the Eulerian description,

$$\begin{aligned}\frac{d}{dt}F^\xi(t) = \frac{d}{dt}f(X^\xi(t), t) &= \nabla f(X^\xi(t), t) \cdot \frac{d}{dt}X^\xi(t) + \frac{\partial f}{\partial t}(X^\xi(t), t) \\ &= \nabla f(X^\xi(t), t) \cdot u(X^\xi(t), t) + \frac{\partial f}{\partial t}(X^\xi(t), t).\end{aligned}$$

Rearranging this, we have

$$\frac{d}{dt}F^\xi(t) = \frac{\partial f}{\partial t}(X^\xi(t), t) + u(X^\xi(t), t) \cdot \nabla f(X^\xi(t), t) \quad \text{for any } \xi \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.6}$$

1.1. **Remark.** It is common to introduce the operator of the material derivative

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + u \cdot \nabla,$$

but I am yet to understand the utility of this besides making things more confusing.

1.2. **Exercise.** Check that (1.5) follows from (1.6) with $F^\xi(t) := X^\xi(t)$ and $f(x, t) := x$.

2. Conservation laws

The conservation of various physical quantities in the evolution of the fluid can be expressed by introducing their density functions first.

By a *measurable physical quantity*[3] we mean a value $\psi(\Omega, t)$ that we can assign to any given (sufficiently nice) subdomain $\Omega \subset \mathbb{R}^n$ at any given time $t \in \mathbb{R}$ and there exists a *density function* $f = f(x, t)$ such that

$$\psi(\Omega, t) = \int_\Omega f(x, t)\, dx. \tag{2.1}$$

Examples are mass, linear momentum, angular momentum and energy, and trivially the volume.

2.1. **Remark.** Note that the density is defined as the amount of the physical quantity per unit volume of the *physical space* and therefore it makes sense to use the Eulerian description in (2.1). Keeping track of the physical quantity for a "chunk" of fluid in the spirit of the Lagrangian description will be handled below by introducing a *moving domain* $\{\Omega_t\}_{t\in\mathbb{R}}$ in (2.2) that contains always the same material points at all times and is in a way "tranported" by the fluid.

[3]Measurable physical quantity is not a standard terminology, but is rather motivated as the analogue of a measure in the measure theory.

Since we will consider only such quantities that are *transported* by the fluid, we want to look at the value of the quantity in a moving domain $\{\Omega_t\}_{t\in\mathbb{R}}$ that contains at each time t the same material points, that is,

$$X^\xi(\tau) \in \Omega_\tau \text{ for some } \xi \in \mathbb{R}^n, \tau \in \mathbb{R} \Rightarrow X^\xi(t) \in \Omega_t \text{ for all } t \in \mathbb{R}.$$

Note that the above implies

$$\Omega_t = \left\{X^\xi(t) : \xi \in \Omega_0\right\} = \{\phi_t(\xi) : \xi \in \Omega_0\} =: \phi_t(\Omega_0). \tag{2.2}$$

Let us therefore consider

$$\psi(\Omega_t, t) = \int_{\Omega_t} f(x,t)\, dx \qquad \text{for } t \in \mathbb{R}.$$

We want to express the time derivative of $t \mapsto \psi(\Omega_t, t)$ in terms of the derivatives of the density function f. The result is given by the following theorem.

2.2. **Theorem** (Transport theorem, Differentiation formula for moving regions)**.**

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} f(x,t)\, dx &= \int_{\Omega_t} \left(\frac{\partial f}{\partial t} + \operatorname{div}(fu)\right)(x,t)\, dx \\ &= \int_{\Omega_t} \frac{\partial f}{\partial t}(x,t)\, dx + \int_{\partial\Omega_t} f(x,t)u(x,t)\cdot \nu_{\partial\Omega_t}(x,t)\, dS\end{aligned} \tag{2.3}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Proof. This is a direct consequence of the change of variables formula. Let us introduce the *Jacobian* $J(\xi,t)$ to be the determinant of the Jacobian matrix of the map ϕ_t at ξ,

$$J(\xi,t) := \det D\phi_t(\xi) \qquad \xi \in \mathbb{R}^n, t \in \mathbb{R}. \tag{2.4}$$

Since $D\phi_0 \equiv I$, by (1.1) we conclude that $J(\xi,t) = \det D\phi_t(\xi) > 0$

We claim that

$$\frac{\partial J}{\partial t}(\xi,t) = J(\xi,t)(\operatorname{div} u)(\phi_t(\xi),t). \tag{2.5}$$

This follows from the formula for the derivative of a determinant of an invertible matrix

$$\frac{d}{dt}\det A(t) = \det A(t)\operatorname{tr}(A(t)^{-1}A'(t)). \tag{2.6}$$

Indeed, let us rewrite (1.5) using $\phi(x,t) = X^\xi(t)$ as

$$u(\phi(\xi,t),t) = \frac{\partial \phi}{\partial t}(\xi,t) \tag{2.7}$$

and differentiate the i-th component of both sides with respect to ξ_j to obtain

$$\sum_k \frac{\partial u_i}{\partial x_k}(\phi(\xi,t),t)\frac{\partial \phi_k}{\partial \xi_j}(\xi,t) = \frac{\partial}{\partial t}\frac{\partial \phi_i}{\partial \xi_j}(\xi,t).$$

This can be written as

$$D_x u(\phi_t(\xi),t) D\phi_t(\xi) = \frac{\partial}{\partial t} D\phi_t(\xi).$$

Since $D\phi_t(\xi)$ is invertible, we can multiply both sides by $(D\phi_t(\xi))^{-1}$, take the trace and use that $\operatorname{tr} AB = \operatorname{tr} BA$ to obtain

$$\operatorname{tr} D_x u(\phi_t(\xi),t) = \operatorname{tr}\left(\left(\frac{\partial}{\partial t}D\phi_t(\xi)\right)(D\phi_t(\xi))^{-1}\right) = \operatorname{tr}\left((D\phi_t(\xi))^{-1}\frac{\partial}{\partial t}D\phi_t(\xi)\right).$$

The left-hand side is just $(\operatorname{div} u)(\phi_t(\xi),t)$, and thus using this in (2.6) we recover (2.5).

By (2.2), the change of variables formula reads

$$\int_{\Omega_t} f(x,t)\,dx = \int_{\Omega_0} f(\phi_t(\xi),t)J(\xi,t)\,d\xi.$$

Since the term on the right-hand side is an integral over a domain independent of t, it is easier to differentiate since we can exchange the integration and differentiation. We have

$$\frac{d}{dt}\int_{\Omega_t} f(x,t)\,dx = \int_{\Omega_0} \frac{\partial}{\partial t}(f(\phi_t(\xi),t)J(\xi,t))\,d\xi. \tag{2.8}$$

By the product rule, chain rule, (2.7) and (2.5), we get

$$\begin{aligned}\frac{\partial}{\partial t}(f(\phi(\xi,t),t)J(\xi,t)) &= \left[\frac{\partial f}{\partial t} + u\cdot\nabla f + f\operatorname{div} u\right](\phi(\xi,t),t)J(\xi,t)\\ &= \left[\frac{\partial f}{\partial t} + \operatorname{div}(fu)\right](\phi(\xi,t),t)J(\xi,t).\end{aligned}$$

Plugging this into (2.8) and using the change of variables back to x and Ω_t, we obtain (2.3). The latter form is a consequence of the divergence theorem. □

2.3. **Remark** ("Physical meaning" of the divergence operator)**.** Volume of a set Ω denoted at time t can be treated as any other physical quantity with the constant density function 1, that is,

$$V(\Omega,t) = \int_\Omega dx \qquad \text{for any } \Omega\subset\mathbb{R}^n, t\in\mathbb{R}.$$

The transport theorem, Theorem 2.2, implies

$$\frac{d}{dt}V(\Omega_t,t) = \int_{\Omega_t} \operatorname{div} u(x,t)\,dx$$

for any moving domain $\{\Omega_t\}_{t\in\mathbb{R}}$ being transported by the fluid. Therefore $\operatorname{div} u$ can be viewed as the *expansion rate of the fluid per unit volume*. An *incompressible* fluid will therefore always have $\operatorname{div} u \equiv 0$.

2.4. **Conservation of mass.** Let us consider the mass $m(\Omega,t)$ of a region of fluid Ω at time $t\in\mathbb{R}$ with density function $\rho=\rho(x,t)$. Since the mass of a given chunk of fluid does not change, given a moving domain $\{\Omega_t\}_{t\in\mathbb{R}}$ being carried by the fluid as in (2.2), the conservation law states that for any $t\in\mathbb{R}$

$$m(\Omega_t,t) = m(\Omega_0,0) \qquad\Leftrightarrow\qquad \frac{d}{dt}m(\Omega_t,t) = 0.$$

The transport theorem, Theorem 2.2, yields

$$0 = \frac{d}{dt}\int_{\Omega_t}\rho(x,t)\,dx = \int_{\Omega_t}\left(\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u)\right)(x,t)\,dx.$$

Since we can choose any $t\in\mathbb{R}$ and any (sufficiently nice) $\Omega_t\subset\mathbb{R}^n$, we may conclude that

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) \equiv 0 \tag{2.9}$$

On the other hand, using the conservation and the change of variables directly, we have

$$\int_{\Omega_0} \rho(x,0)\,dx = \int_{\Omega_t} \rho(x,t)\,dx = \int_{\Omega_0} \rho(X^\xi(t),t)J(\xi,t)\,d\xi,$$

where the Jacobian J was introduced in (2.4). Since Ω_0 and t are arbitrary we have a formula for the density in the Lagrangian description

$$R^\xi(t) = \rho(X^\xi(t),t) = \frac{\rho(\xi,0)}{J(\xi,t)} = \frac{R^\xi(0)}{J(\xi,t)}. \tag{2.10}$$

The Jacobian $J(\xi,t)$ therefore represents the local compression ratio of the fluid over time t. Recall that by the assumption (1.1) $J(\xi,t) > 0$ and therefore the above is always a well-defined finite quantity.

2.5. **Exercise.** Derive (2.9) directly by taking the time derivative of $R^\xi(t)$ in (2.10) and using (2.5).

2.6. **Remark.** Using the divergence theorem (integration by parts), we get for any $\Omega \subset \mathbb{R}^n$ and $t \in \mathbb{R}$

$$0 = \int_\Omega \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u)\right)(x,t)\,dx = \int_\Omega \frac{\partial \rho}{\partial t}(x,t)\,dx + \int_{\partial\Omega} \rho(x,t)u(x,t)\cdot \nu_{\partial\Omega}(x,t)\,dS.$$

From this we see that ρu is the *flux* of mass.

2.7. **Conservation of momentum.** The linear momentum[4] $M(\Omega,t)$ has density function $f(x,t) = \rho(x,t)u(x,t)$ by definition. Newton's law states that the momentum of a chunk of fluid changes by the action of a force, that is, the momentum of the fluid contained in the moving domain $\{\Omega_t\}_{t\in\mathbb{R}}$ satisfies

$$\frac{d}{dt}M(\Omega_t,t) = \Phi(\Omega_t,t), \tag{2.11}$$

where $\Phi(\Omega_t,t)$ is the total force acting on the chunk at time t. This total force can be split into volume (body) forces like gravity acting per unit volume with unit density $b(x,t)$ and surface forces like pressure that represent the interaction with the surrounding fluid with unit density $s(x,t,\nu)$ per unit surface area with outer unit normal vector ν,

$$\Phi(\Omega,t) := \int_\Omega b(x,t)\,dx + \int_{\partial\Omega} s(x,t,\nu_{\partial\Omega}(x,t))\,dS.$$

Let us consider the components of the momentum separately. For the component M_i, $1 \le i \le n$, the transport theorem, Theorem 2.2, yields

$$\frac{d}{dt}M_i(\Omega_t,t) = \frac{d}{dt}\int_{\Omega_t} \rho(x,t)u_i(x,t)\,dx = \int_{\Omega_t}\left(\frac{\partial(\rho u_i)}{\partial t} + \operatorname{div}(\rho u_i u)\right)(x,t)\,dx.$$

Since $t \in \mathbb{R}$ and Ω_t are arbitrary, from (2.11) we get for any $\Omega \subset \mathbb{R}^n$, $t \in \mathbb{R}$ and $i = 1,\dots,n$

$$\int_\Omega \left(\frac{\partial(\rho u_i)}{\partial t} + \operatorname{div}(\rho u_i u) - b_i\right)(x,t)\,dx = \int_{\partial\Omega} s_i(x,t,\nu_{\partial\Omega}(x,t))\,dS. \tag{2.12}$$

If we can show that $s(x,t,p)$ is linear in p, we get the standard form of this conservation law. One can as well assume that $s(x,t,p) = \sigma(x,t)p$ for some matrix

[4]Note that for M as a transported quantity there is no notion of Eulerian vs. Lagrangian description. Uppercase M is just the standard notation for linear momentum.

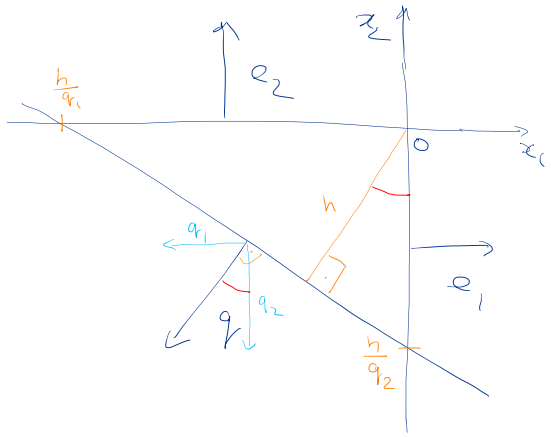

FIGURE 1. Domain Ω_h.

function σ, but we will show that there is a *mathematical* reason why this is the only possible form.

2.8. **Theorem** (Cauchy's theorem)**.** *If s is a continuous function, then there exists a matrix-valued function $\sigma = \sigma(x,t)$ such that $s(x,t,q) = \sigma(x,t)q$ for all $q \in \mathbb{R}^n$, $|q| = 1$.*

Proof. We will show this using a scaling argument with the help of (2.12). Note that the term on the left-hand side in (2.12) is proportional to the volume of Ω, while the right-hand side is proportional to the surface area of $\partial\Omega$. Since $A(\partial\Omega) \geq C|\Omega|^{\frac{n-1}{n}}$ by the isoperimetric inequality, we see that the surface area will be much bigger than the volume for small domains. The only way this is possible is if there is some cancellation in the integrand on the right-hand side. We will see that this cancellation happens only if the integrand $s(x,t,\nu)$ is linear in ν.

For simplicity, we assume that $x = 0$. Let us first consider a vector $q \in \mathbb{R}^n$ with size $|q| = 1$ such that all its components are negative, $q_i < 0$ for all $i = 1, \ldots, n$. For h define the domain

$$\Omega_h := \{x : x_i < 0 \text{ for } i = 1, \ldots, n,\ x \cdot q < h\}$$

Ω_h is the n-dimensional simplex with vertices 0, $\frac{h}{q_1}e_1$, $\ldots$, $\frac{h}{q_n}e_n$, see Figure 1. It has $n+1$ flat sides denoted Γ_0, Γ_1, $\ldots$, Γ_n opposite to each of the above vertices with normal vectors q, e_1, $\ldots$, e_n. The volume Ω_h is given as $\frac{h^n}{n!\prod_i |q_i|}$. The area $A(\Gamma_0)$ of the flat side Γ_0 opposite to the vertex 0 is $\frac{h^{n-1}}{(n-1)!\prod_i |q_i|}$, and the area $A(\Gamma_j)$ of the flat side Γ_j opposite to the vertex $\frac{h}{q_j}e_j$ it is $\frac{q_j h^{n-1}}{(n-1)!\prod_i |q_i|}$. In particular,

$$\frac{A(\Gamma_0)}{A(\partial\Omega_h))} = \frac{1}{1+\sum_i |q_i|}, \qquad \frac{A(\Gamma_j)}{A(\partial\Omega_h))} = \frac{q_j}{1+\sum_i |q_i|}. \tag{2.13}$$

Let us denote the left-hand side in (2.12) for $\Omega = \Omega_h$ by $\mu_i(h)$. Then we can write (2.12) as

$$\int_{\partial\Omega_h} s(x,t,\nu_{\partial\Omega_h}(x,t))\,dS = \mu(h).$$

Since the integrand on the left-hand side of (2.12) is continuous, there exists $K > 0$ such that $|\mu(h)| \le K|\Omega_h| \le Ch^n$.

On the other hand, we can compute

$$\int_{\partial\Omega_h} s(x,t,\nu_{\partial\Omega_h}(x,t))\,dS = \int_{\Gamma_0} s(x,t,q)\,dS + \sum_{i=1}^{n}\int_{\Gamma_i} s(x,t,e_i)\,dS.$$

If we divide the above equation by the area of the boundary $A(\partial\Omega_h)$, we get

$$\frac{1}{A(\partial\Omega_h)}\left(\int_{\Gamma_0} s(x,t,q)\,dS + \sum_{i=1}^{n}\int_{\Gamma_i} s(x,t,e_i)\,dS\right) = \frac{\mu(h)}{A(\Omega_h)}.$$

Sending $h \to 0$ using the continuity of the integrands and observing that the right-hand side converges to 0, and using (2.13), we obtain

$$s(0,t,q) + \sum_{i=1}^{n} s(0,t,e_i)|q_i| = 0$$

In particular, we see that

$$s(0,t,q) = -\sum_{i=1}^{n} s(0,t,e_i)|q_i| = \sum_{i=1}^{n} s(0,t,e_i)q_i.$$

We can follow an analogous argument if q_i have other than negative signs to obtain in general for q with $q_i \neq 0$ for all $i = 1,\dots,n$,

$$s(0,t,q) = \sum_{i=1}^{n} -(\operatorname{sign} q_i)s(0,t,-(\operatorname{sign} q_i)e_i)q_i.$$

A similar argument, using a box with one side much shorter than the other $n-1$, for example $\Omega_h = \{x : |x_i| \le h,\ i = 1,\dots,n-1,\ |x_n| \le h^2\}$ for the case e_n, yields that s is odd in ν,

$$s(x,t,e_i) = -s(x,t,-e_i) \qquad \text{for all } i = 1,\dots,n.$$

Therefore we can write

$$s(x,t,q) = \sum_{i=1}^{n} s(x,t,e_i)q_i.$$

We define

$$\sigma_{ij}(x,t) = s_i(x,t,e_j) \qquad \text{for } i,j = 1,\dots,n.$$

□

References

[EG] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Revised edition, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015. MR3409135