LARGE-TIME BEHAVIOR OF ONE-PHASE STEFAN-TYPE PROBLEMS WITH ANISOTROPIC DIFFUSION IN PERIODIC MEDIA

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ABSTRACT. We study the large-time behavior of solutions of a one-phase Stefan-type problem with anisotropic diffusion in periodic media on an exterior domain in a dimension $n \geq 3$. By a rescaling transformation that matches the expansion of the free boundary, we deduce the homogenization of the free boundary velocity together with the homogenization of the anisotropic operator. Moreover, we obtain the convergence of the rescaled solution to the solution of the homogenized Hele-Shaw-type problem with a point source and the convergence of the rescaled free boundary to a self-similar profile with respect to the Hausdorff distance.

1. INTRODUCTION

We analyze the behavior of an anisotropic one-phase Stefan-type problem with periodic coefficients on an exterior domain in a dimension $n \geq 3$. Our purpose is to investigate the asymptotic behavior of the solution of the problem (1.1) below and its free boundary as time $t \to \infty$. The results in this paper are the generalizations of our previous work in [38] for the isotropic case.

We consider a compact set $K \subset \mathbb{R}^n$ that represents a source. We assume that $0 \in \operatorname{int} K$ and K has a sufficiently regular boundary, $\partial K \in C^{1,1}$ for example. The one-phase Stefan-type problem with anisotropic diffusion is to find a function $v(x,t) : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ satisfying

$$
(1.1) \quad
\begin{cases}
v_t - D_i(a_{ij} D_j v) = 0 & \text{in } \{v > 0\} \setminus K, \\
\qquad\quad v = 1 & \text{on } K, \\
\dfrac{v_t}{|Dv|} = g\, a_{ij}\, D_j v\, \nu_i & \text{on } \partial\{v > 0\}, \\
\quad v(x,0) = v_0 & \text{on } \mathbb{R}^n,
\end{cases}
$$

where D is the space gradient, D_i is the partial derivative with respect to x_i, v_t is the partial derivative of v with respect to time variable t and $\nu = \nu(x,t)$ is the inward spatial unit normal vector of $\partial\{v > 0\}$ at a point (x,t). Here we use the Einstein summation convention.

The Stefan problem is a free boundary problem of parabolic type for phase transitions, typically describing the melting of ice in contact with a region of water. Here we consider the one-phase problem, where the temperature is assumed to be

maintained at 0 in one of the phases. We prescribe the Dirichlet boundary data 1 on the fixed source K and an initial temperature distribution v_0. Note that the results in this paper apply to a more general time-independent positive fixed boundary data, the constant function 1 is taken only to simplify the notation. We also specify an inhomogeneous medium with the latent heat of phase transition $L(x) = \frac{1}{g(x)}$ and an anisotropic diffusion with the thermal conductivity coefficients given by $a_{ij}(x)$. The unknowns here are the temperature distribution v and the phase interface $\partial\{v > 0\}$, which is the so-called free boundary. Since the free boundary is a level set of v, the outward normal velocity of the moving interface is given by $\frac{v_t}{|Dv|}$. The free boundary condition thus says that the interface moves outward with the velocity $g a_{ij} D_j v \nu_i$ in the normal direction. Note that we can also rewrite the free boundary condition as

$$(1.2) \qquad v_t = g \, a_{ij} \, D_j v D_i v.$$

Throughout this paper, we will consider the problem under the following assumptions. The matrix $A(x) = (a_{ij}(x))$ is assumed to be symmetric, bounded, and uniformly elliptic, i.e., there exits some positive constants α and β such that

$$(1.3) \qquad \alpha|\xi|^2 \le a_{ij}(x)\,\xi_i\xi_j \le \beta|\xi|^2 \quad \text{ for all } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n.$$

Moreover, we are interested in the problems with highly oscillating coefficients that guarantee an averaging behavior in the scaling limit, in particular,

$$(1.4) \qquad \begin{aligned} &a_{ij} \text{ and } g \text{ are } \mathbb{Z}^n\text{-periodic Lipschitz functions in } \mathbb{R}^n, \\ &m \le g \le M \text{ for some positive constants } m \text{ and } M. \end{aligned}$$

From the ellipticity (1.3) and the boundedness of g, we also have

$$(1.5) \qquad m\alpha|\xi|^2 \le g(x)a_{ij}(x)\xi_i\xi_j \le M\beta|\xi|^2 \text{ for all } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n.$$

Furthermore, during almost the whole paper, the initial data is assumed to satisfy

$$(1.6) \qquad \begin{aligned} &v_0 \in C^2(\overline{\Omega_0 \setminus K}), v_0 > 0 \text{ in } \Omega_0, v_0 = 0 \text{ on } \Omega_0^{\mathsf{c}} := \mathbb{R}^n \setminus \Omega_0, \text{ and } v_0 = 1 \text{ on } K, \\ &|Dv_0| \ne 0 \text{ on } \partial\Omega_0, \text{ for some bounded domain } \Omega_0 \supset K. \end{aligned}$$

Here we use a stronger regularity of the initial data than the general requirement to guarantee the well-posedness of the Stefan problem (1.1) (see [19, 28]) and the coincidence of weak and viscosity solutions used in our work (see [30]). However, our convergence results rely on a crucial weak monotonicity (4.5) which holds provided the initial data satisfies (1.6). Nevertheless, the asymptotic limit in Theorem 1.1 is independent of the initial data. Therefore we are able to apply the results for more general initial data. In particular, it is sufficient if the initial data guarantees the existence of the (weak) solution satisfying the comparison principle, and we can approximate the initial data from below and from above by regular data satisfying (1.6). As the classical problem [38], $v_0 \in C(\mathbb{R}^n), v_0 = 1$ on $K, v_0 \ge 0, \operatorname{supp} v_0$ compact is enough.

A global classical solution of the Stefan problem (1.1) is not expected to exist due to the singularities of the free boundary, which might occur in finite time. This motivated the introduction of a generalized solution of this problem. In this paper, we will use the notions of weak solutions and viscosity solutions. The notion of weak solutions is more classical, defined by taking the integral in time of the classical solution v and looking at the equation that the new function $u(x,t) := \int_0^t v(x,s)\,ds$

satisfies. It turns out that if v is sufficiently regular, then $u(\cdot, t)$ solves the obstacle problem (see [4, 13, 19, 14, 41, 43, 42])

(1.7)
$$\begin{cases} u(\cdot, t) \in \mathcal{K}(t), \\ (u_t - D_i(a_{ij} D_j u)) (\varphi - u) \geq f(\varphi - u) \text{ a.e } (x, t) \text{ for any } \varphi \in \mathcal{K}(t), \end{cases}$$

where $\mathcal{K}(t)$ is a suitable admissible functional space (see Section 2.1.1) and f is

(1.8)
$$f(x) = \begin{cases} v_0(x), & v_0(x) > 0, \\ -\dfrac{1}{g(x)}, & v_0(x) = 0. \end{cases}$$

This formulation interprets the Stefan problem as a fixed domain problem and allows us to apply the well-known results in the general variational inequality theory. Indeed, the obstacle problem (1.7) has a global unique solution u for the initial data (1.6). If the corresponding time derivative $v = u_t$ exists, it is called a *weak solution* of the Stefan problem (1.1). Moreover, the homogenization of this problem was also studied using the homogenization theory of variational inequalities, see [40, 29, 30]. In a different approach based on the comparison principle structure, Kim introduced the notion of viscosity solutions of the Stefan problem as well as proved the global existence and uniqueness results in [28], which were later generalized to the two-phase Stefan problem in [31]. The analysis of viscosity solutions relies on the comparison principle and pointwise arguments, which are more suitable for the study of the behavior of the free boundaries. The notions of weak and viscosity solutions were first introduced for the classical homogeneous isotropic Stefan problem where $g(x) = 1$ and the parabolic operator is the simple heat operator, however, it is natural to define the same notions for the Stefan problem (1.1) and obtain the analogous results as observed in [40, 30]. Moreover, the notion of viscosity solutions is also applicable for more general, fully nonlinear parabolic operators and boundary velocity laws since it does not require the variational structure. In [30], Kim and Mellet showed that the weak and the viscosity solutions of (1.1) coincide whenever the weak solution exists. We will use the strengths of both weak and viscosity solutions to study (1.1).

Among the first results on the asymptotic and large time behavior of solutions of the one-phase Stefan problem on an exterior domain was the work of Matano [33] for the classical homogeneous isotropic Stefan problem in dimensions $n \geq 3$. He showed that in this setting, any weak solution eventually becomes classical after a finite time and that the shape of the free boundary approaches a sphere of radius $Ct^{\frac{1}{n}}$ as $t \to \infty$. Note that in our case of an inhomogeneous free boundary velocity we cannot expect the solution to become classical even for large times. In [39], Quirós and Vázques extended the results of [33] to the case $n = 2$ and showed the large-time convergence of the rescaled weak solution of the one-phase Stefan problem to the self-similar solution of the Hele-Shaw problem with a point source. The related Hele-Shaw problem is also called the quasi-stationary Stefan problem, where the heat operator is replaced by the Laplace operator. It typically models the flow of a viscous fluid injected in between two parallel plates which form the so-called Hele-Shaw cell, or the flow in porous media. The global stability of steady solutions of the classical Stefan problem on the full space was established in the recent work of Hadžić and Shkoller [24, 25], and further developed for the two-phase Stefan problem in [23].

The homogenization of the one-phase Stefan problem (1.1) and the Hele-Shaw problem in both periodic and random media was obtained by Rodrigues in [40] and later by Kim and Mellet in [29, 30]. See also the work on the homogenization of the two-phase Stefan problem by Bossavit and Damlamian [6]. Dealing directly with the large-time behavior of the solutions on an exterior domain in inhomogeneous media, the work of the first author in [36], and then the joint work of both authors in [38] showed the convergence of an appropriate rescaling of solutions of both models to the self-similar solution of the Hele-Shaw problem with a point source. The rescaled free boundary was shown to uniformly approach a sphere.

In this paper, we extend the previous results in [38] to the anisotropic case, where the heat operator is replaced by a more general linear parabolic operator of divergence form. This was indeed the setting considered in [40, 30] for the homogenization problems. In this setting, the variational structure is preserved, thus we are still able to use the notions of weak solutions as well as viscosity solutions and their coincidence. However, the main difficulties come from the loss of radially symmetric solutions which were used as barriers in the isotropic case and the homogenization problems appear not only for the velocity law but also for the elliptic operators. To overcome the first difficulty, we will construct barriers for our problem from the fundamental solution of the corresponding elliptic equation in a divergence form. Unfortunately, even though the unique fundamental solution of this elliptic equation exists for $n \geq 2$, its asymptotic behavior in dimension $n = 2$ and dimensions $n \geq 3$ are significantly different. Moreover, we need to use the gradient estimate (2.7) for the fundamental solution, which only holds for the periodic structure. Therefore, we will limit our consideration to periodic media and dimension $n \geq 3$. Following [39, 36, 38], we use the rescaling of solutions as

$$v^\lambda(x,t) := \lambda^{\frac{n-2}{n}} v(\lambda^{\frac{1}{n}} x, \lambda t), \quad u^\lambda(x,t) := \lambda^{-\frac{2}{n}} u(\lambda^{\frac{1}{n}} x, \lambda t), \quad \lambda > 0.$$

Using this rescaling we can deduce the uniform convergence of the rescaled solution to a limit function away from the origin. In the limit, the fixed domain K shrinks to the origin due to the rescaling, and the rescaled solutions develop a singularity at the origin as $\lambda \to \infty$. Moreover, in a periodic setting, the elliptic operator and velocity law should homogenize as $\lambda \to \infty$, and therefore heuristically the limit function should be the self-similar solution of the Hele-Shaw-type problem with a point source

$$(1.9) \qquad \begin{cases} -q_{ij} D_{ij} v = C\delta & \text{in } \{v > 0\}, \\ \quad v_t = \dfrac{1}{\left\langle \frac{1}{g} \right\rangle} q_{ij} D_i v D_j v & \text{on } \partial\{v > 0\}, \\ v(\cdot, 0) = 0, \end{cases}$$

where δ is the Dirac δ-function, (q_{ij}) is a constant symmetric positive definite matrix depending only on a_{ij}, C is a constant depending on K, n, q_{ij} and the boundary data 1, and the constant $\left\langle \frac{1}{g} \right\rangle$ is the average value of the latent heat $L(x) = \frac{1}{g(x)}$. Similarly, the limit variational solution should satisfy the corresponding limit obstacle problem.

The first main result of this paper, Theorem 3.1, is the locally uniform convergence of the rescaled variational solution to the solution of the limit obstacle

problem. Using the constructed barriers, we are able to prove that the limit function has the correct singularity as $|x| \to 0$. Moreover, the barriers also give the same growth rate for the free boundary as in the isotropic case. That is, the free boundary expands with the rate $t^{\frac{1}{n}}$ when t is large enough. The aim is then to prove the homogenization effects of the rescaling to our problem. The shrinking of the fixed domain K in the rescaling also makes our current situation slightly different from the standard classical homogenization problem of variational inequalities, where the domain and the boundary condition are usually fixed. In addition, we also need to show that the rescaled parabolic operator becomes elliptic when $\lambda \to 0$. We will use the notion of the Γ-convergence introduced by De Giorgi and homogenization techniques developed by Dal Maso and Modica in [9, 10, 11]. The issue here is that we need to modify the Γ-convergence sequence in order to use the integration by parts formula for the variational inequality. This will be done with the help of a cut-off function and the *fundamental estimate*, Definition 3.6, for the Γ-convergence. Note that these techniques are applicable not only for the periodic case but also for the random case, thus we expect to extend our results to the problem in random media in the future.

As the last step, we will use the coincidence of the weak and viscosity solution of the problem (1.1) and the viscosity arguments to obtain the uniform convergence of the rescaled viscosity solution and its free boundary to the asymptotic profile in the second main result, Theorem 4.1. Fortunately, all the viscosity arguments of the isotropic case can be adapted for the anisotropic case. Therefore the proof is similar to the proof of [38, Theorem 4.2], where we make use of a weak monotonicity (4.5) together with the comparison principle. An important point in the proof of [38, Theorem 4.2] is that we need to apply Harnack's inequality for a parabolic equation which becomes elliptic in the limit. Here we can proceed as in the isotropic case since the rescaled elliptic operator does not change the constant in Harnack's inequality. As the arguments require only simple modifications, we will skip the proofs of some lemmas and refer to [38] for more details.

In summary, we will show the following theorem.

Theorem 1.1. *The rescaled viscosity solution v^λ of the Stefan-type problem (1.1) converges locally uniformly to the unique self-similar solution V of the Hele-Shaw type problem (1.9) in $(\mathbb{R}^n \setminus \{0\}) \times [0, \infty)$ as $\lambda \to \infty$, where (q_{ij}) is a constant symmetric positive definite matrix depending only on a_{ij}, C depends only on q_{ij}, n, the set K and the boundary data 1. Moreover, the rescaled free boundary $\partial\{(x, t) : v^\lambda(x, t) > 0\}$ converges to $\partial\{(x, t) : V(x, t) > 0\}$ locally uniformly with respect to the Hausdorff distance.*

As mentioned above, almost all of the arguments in our recent work hold for stationary ergodic random case. However, in this situation, we lose a very important pointwise gradient estimate (2.7) for the fundamental solution of the corresponding elliptic equation to construct the barriers. In fact, for non-periodic coefficients, even though the optimal bounds for the gradient continue to hold for a bounded domain, they cannot hold in the large scale when $|x - y| \to \infty$. The results in [34, 22] tell us that for random stationary coefficients satisfying a logarithmic Sobolev inequality we have similar bounds for the gradient in local square average forms. This result cannot be upgraded to the pointwise bounds since there is no regularity to control the square average integral as in [22, Remark 3.7]. However, it suggests the possibility to modify our approach to the random case. Another question is the

extension of the present results to the dimension $n = 2$. Since the unique (up to an addition of a constant) fundamental solution of the corresponding elliptic equation exists and the gradient estimates also hold in the two-dimensional case, we expect to obtain analogous results as in this paper. The essential reason that it remains open is the lack of a homogenization result for the fundamental solution (Green's function) in two dimensions, which is of an independent interest. This issue is under investigation by the authors.

The structure of the paper is as follows: The definitions and the well-known results for weak and viscosity solutions are recalled in Section 2. We also review some basic known facts about the fundamental solution of the corresponding elliptic equation. The rescaling is introduced and we discuss the convergence of the fundamental solution in the rescaling limit. The core of this section is the construction of a subsolution and a supersolution of the Stefan problem (1.1) in Section 2.4. Moreover, we formulate the limit problems before giving the proofs of the main results in the later sections. Section 3 is our main contribution, where we prove the locally uniform convergence of the rescaled variational solutions. In Section 4, we deal with the locally uniform convergence of the rescaled viscosity solutions and their free boundaries.

Notation. We will use the following notations throughout this paper. For a set A, A^{c} is its complement. Given a nonnegative function v, we will denote its positive set and free boundary as

$$\Omega(v) := \{(x,t) : v(x,t) > 0\}, \qquad \Gamma(v) := \partial\Omega(v),$$

and for a fixed time t,

$$\Omega_t(v) := \{x : v(x,t) > 0\}, \qquad \Gamma_t(v) := \partial\Omega_t(v).$$

We will denote the elliptic operator of divergence form and its rescaling as

$$(1.10) \qquad \mathcal{L}u = D_j(a_{ij}D_iu), \qquad \mathcal{L}^\lambda u = D_j(a_{ij}(\lambda^{\frac{1}{n}}x)D_iu), \quad \lambda > 0.$$

We will also make use of the bilinear forms in $H^1(\Omega)$ and the inner product in $L^2(\Omega)$ as

$$a_\Omega(u,v) = \int_\Omega a_{ij}D_iuD_jv\,dx, \quad a_\Omega^\lambda(u,v) = \int_\Omega a_{ij}(\lambda^{\frac{1}{n}}x)D_iuD_jv\,dx, \quad \lambda > 0$$

$$q_\Omega(u,v) = \int_\Omega q_{ij}D_iuD_jv\,dx, \quad \langle u,v \rangle_\Omega = \int_\Omega uv\,dx,$$

where q_{ij} are the constant coefficients of the homogenized operator. We omit the set Ω in the notation if $\Omega = \mathbb{R}^n$.

2. Preliminaries

2.1. **Notion of solutions.**

2.1.1. *Weak solutions.* As for the classical one-phase Stefan problem, we will define the weak solutions of (1.1) using the corresponding variational problem given in [19,30]. Let $B = B_R(0)$, $D = B \backslash K$ for some fixed $R \gg 1$. Find $u \in L^2(0, T; H^2(D))$ such that $u_t \in L^2(0, T; L^2(D))$ and

$$(2.1) \quad \begin{cases} u(\cdot, t) \in \mathcal{K}(t), & 0 < t < T, \\ (u_t - \mathcal{L}u)(\varphi - u) \geq f(\varphi - u), & \text{a.e } (x, t) \in D \times (0, T) \text{ for any } \varphi \in \mathcal{K}(t), \\ u(x, 0) = 0 \text{ in } D, \end{cases}$$

where the admissible set $\mathcal{K}(t)$ is

$$\mathcal{K}(t) = \{\varphi \in H^1(D), \varphi \geq 0, \varphi = 0 \text{ on } \partial B, \varphi = t \text{ on } K\}$$

and

$$f(x) := \begin{cases} v_0(x) & \text{for } x \in \Omega_0, \\ -\dfrac{1}{g(x)} & \text{for } x \in \Omega_0^{\mathsf{c}}. \end{cases}$$

We use the standard notation H^k and $W^{k,p}$ for Sobolev spaces.

Following [19], if $v(x, t)$ is a classical solution of the Stefan problem (1.1) in $D \times (0, T)$ and R is sufficiently large depending on T, then the function $u(x, t) := \int_0^t v(x, s) \, ds$ solves (2.1). On the other hand, it was shown in [19, 41] that the variational problem (2.1) is well-posed for initial data v_0 satisfying (1.6).

Theorem 2.1 (Existence and uniqueness for the variational problem). *If v_0 satisfies (1.6), then the problem (2.1) has a unique solution satisfying*

$$u \in L^\infty(0, T; W^{2,p}(D)), \qquad 1 \leq p \leq \infty,$$
$$u_t \in L^\infty(D \times (0, T)),$$

and

$$\begin{cases} u_t - \mathcal{L}u \geq f, & u \geq 0, \\ u(u_t - \mathcal{L}u - f) = 0 \end{cases} \quad a.e \text{ in } D \times (0, \infty).$$

Thus we define the *weak solution* of the Stefan problem (1.1) as the time derivative u_t of the solution u of (2.1). Note that as in [30, Lemma 3.6], the solution does not depend on the choice of R if R is sufficiently large.

We list here some useful properties of the weak solutions for later use, see [19, 41, 30].

Proposition 2.2. *The unique solution u of (2.1) satisfies $u_t \in C(\overline{D} \times [0, T])$ and*

$$0 \leq u_t \leq C \qquad a.e. \ D \times (0, T),$$

where C is a constant depending on f. In particular, u is Lipschitz with respect to t and u is $C^\alpha(D)$ with respect to x for all $\alpha \in (0, 1)$. Furthermore, if $0 \leq t < s \leq T$, then $u(\cdot, t) < u(\cdot, s)$ in $\Omega_s(u)$ and also $\Omega_0 \subset \Omega_t(u) \subset \Omega_s(u)$.

Proof. Let us only give a remark on $u_t \in C(\overline{D} \times [0, T])$, the rest is standard following the arguments in the cited papers and the elliptic regularity theory. The regularity of weak solutions and their free boundaries was studied by many authors, see [7, 8, 27, 12, 48, 44]. If $a_{ij} = \delta_{ij}$, the temperature u_t is continuous in $\mathbb{R}^n \times [0, \infty)$ due to the result of Caffarelli and Friedman [8]. By a change of coordinates, the continuity of u_t can also be obtained when the coefficients are constants. Using a different approach for more general singular parabolic equations, Di Benedetto [12], Ziemer [48] and

Sacks [44] showed that the continuity also holds in the case $a_{ij} = a_{ij}(x)$ satisfying (1.3). Note that the assumptions on the L^∞-bound of the weak solution $v = u_t$ and the L^2-bounds of its derivatives as in [48, 12, 44] are guaranteed by Proposition 2.2 above and [19, Theorem 3] or [17, Corollary 2, Theorem 4].   □

Lemma 2.3 (Comparison principle for weak solutions)**.** *Suppose that* $f \leq \hat{f}$*. Let* $u, \hat{u}$ *be solutions of (2.1) for respective* $f, \hat{f}$*. Then* $u \leq \hat{u}$ *and*

$$v \equiv u_t \leq \hat{u}_t \equiv \hat{v}.$$

2.1.2. *Viscosity solutions.* Generalized solutions of the Stefan problem (1.1) can also be defined via the comparison principle, leading to the viscosity solutions introduced in [28]. In the following, Q is the space-time cylinder $Q := (\mathbb{R}^n \setminus K) \times [0, \infty)$.

Definition 2.4. A nonnegative upper semicontinuous function $v = v(x, t)$ defined in Q is a viscosity subsolution of (1.1) if:

a) For all $T \in (0, \infty)$, $\overline{\Omega(v)} \cap \{t \leq T\} \cap Q \subset \overline{\Omega(v) \cap \{t < T\}}$.

b) For every $\phi \in C^{2,1}_{x,t}(Q)$ such that $v - \phi$ has a local maximum in $\overline{\Omega(v)} \cap \{t \leq t_0\} \cap Q$ at (x_0, t_0), the following holds:

   i) If $v(x_0, t_0) > 0$, then $(\phi_t - \mathcal{L}\phi)(x_0, t_0) \leq 0$.

   ii) If $(x_0, t_0) \in \Gamma(v), |D\phi(x_0, t_0)| \neq 0$ and $(\phi_t - \mathcal{L}\phi)(x_0, t_0) > 0$, then

(2.2) $(\phi_t - g a_{ij} D_j \phi \nu_i |D\phi|)(x_0, t_0) \leq 0,$

   where ν is inward spatial unit normal vector of $\partial\{v > 0\}$.

Analogously, a nonnegative lower semicontinuous function $v(x, t)$ defined in Q is a viscosity supersolution if (b) holds with maximum replaced by minimum, and with inequalities reversed in the tests for ϕ in (i–ii). We do not need to require (a).

Remark 2.5. As in [37, Remark 2.4], the condition a) guarantees the continuous expansion of the support of the subsolution v, which prevents "bubbles" closing up, that is, it prevents v becoming instantly positive in the whole space or in an open set surrounded by a positive phase.

Definition 2.6. A viscosity subsolution of (1.1) in Q is a viscosity subsolution of (1.1) in Q with initial data v_0 and boundary data 1 if:

a) v is upper semicontinuous in $\bar{Q}, v = v_0$ at $t = 0$ and $v \leq 1$ on Γ,

b) $\overline{\Omega(v)} \cap \{t = 0\} = \overline{\{x : v_0(x) > 0\}} \times \{0\}$.

A viscosity supersolution is defined analogously by requiring (a) with v lower semicontinuous and $v \geq 1$ on Γ. We do not need to require (b).

A viscosity solution is both a subsolution and a supersolution:

Definition 2.7. The function $v(x, t)$ is a viscosity solution of (1.1) in Q (with initial data v_0 and boundary data 1) if v is a viscosity supersolution and $v^\star$ is a viscosity subsolution of (1.1) in Q (with initial data v_0 and boundary data 1). Here $v^\star$ is the upper semicontinuous envelopes of v define by

$$w^\star(x, t) := \limsup_{(y,s) \to (x,t)} w(y, s).$$

The notion of viscosity solutions of the classical Stefan problem was first introduced in [28]. It was generalized to the problem (1.1) in [30] including a comparison

principle for "strictly separated" initial data. More importantly, in [30] the authors proved the coincidence of weak and viscosity solutions which will be used as a crucial tool in our work.

Theorem 2.8 (cf. [30, Theorem 3.1]). *Assume that v_0 satisfies (1.6). Let $u(x,t)$ be the unique solution of (2.1) in $B \times [0, T]$ and let $v(x, t)$ be the solution of*

(2.3)
$$\begin{cases} v_t - \mathcal{L}v = 0 & in\ \Omega(u) \setminus K, \\ \quad v = 0 & on\ \Gamma(u), \\ \quad v = 1 & in\ K, \\ v(x,0) = v_0(x). \end{cases}$$

Then $v(x, t)$ is a viscosity solution of (1.1) in $B \times [0, T]$ with initial data $v(x, 0) = v_0(x)$, and $u(x, t) = \int_0^t v(x, s)\, ds$.

By the coincidence of weak and viscosity solutions, we have a more general comparison principle as follows.

Lemma 2.9 (cf. [30, Corollary 3.12]). *Let v^1 and v^2 be, respectively, a viscosity subsolution and supersolution of the Stefan problem (1.1) with continuous initial data $v_0^1 \leq v_0^2$ and boundary data 1. In addition, suppose that v_0^1 (or v_0^2) satisfies condition (1.6). Then $v_\star^1 \leq v^2$ and $v^1 \leq (v^2)^\star$ in $\mathbb{R}^n \setminus K \times [0, \infty)$.*

Remark 2.10. We first note that a classical subsolution (supersolution) of (1.1) is also a viscosity subsolution (supersolution) of (1.1) in Q with initial data v_0 and boundary data 1 by standard arguments.

Moreover, if $\Omega(u)$ is not smooth, we need to understand the solution of (2.3) as the one given by Perron's method as

$$v = \sup\{w \mid w_t - \Delta w \leq 0 \text{ in } \Omega(u), w \leq 0 \text{ on } \Gamma(u), w \leq 1 \text{ in } K, w(x, 0) \leq v_0(x)\},$$

which allows v to be discontinuous on $\Gamma(u)$.

2.2. **The fundamental solution of a linear elliptic equation.** In this section, we will recall some important facts about the fundamental solution of the self-adjoint uniformly elliptic second order linear equation of divergence form

(2.4)
$$- \mathcal{L}u = 0,$$

in dimension $n \geq 3$, where $\mathcal{L}$ was defined in (1.10) and $a_{ij}(x)$ satisfy (1.3) and (1.4). This fundamental solution will be used to construct barriers for the Stefan problem (1.1).

We define the fundamental solution of (2.4) as Green's function in the whole space following [32, 1].

Definition 2.11. We say that $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the fundamental solution (Green's function) of (2.4) if $G(\cdot, y)$ is the weak (distributional) solution of $-\mathcal{L}G(\cdot, y) = \blacksquare$ δ_y, where δ_y is the Dirac measure at y, i.e.,

$$\int_{\mathbb{R}^n} a_{ij} D_j G(\cdot, y) D_i \varphi\, dx = \varphi(y), \quad \forall y \in \mathbb{R}^n, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n),$$

and $\lim_{|x-y| \to \infty} G(x, y) = 0$.

The existence and uniqueness of the fundamental solution were given by the remark following [32, Corollary 7.1] or more precisely by [1, Theorem 1].

Theorem 2.12 (cf. [1, Theorem 1]). *Assume that $n \geq 3$, $a_{ij}(x)$ satisfy (1.3) and (1.4). Then, there exists a unique fundamental solution G of (2.4) such that $G(\cdot, y) \in H^1_{loc}(\mathbb{R}^n \setminus \{y\}) \cap W^{1,p}_{loc}(\mathbb{R}^n), p < \frac{n}{n-1}$, and for some constant $C > 0$ we have*

$$(2.5) \qquad C^{-1}|x - y|^{2-n} \leq G(x, y) \leq C|x - y|^{2-n}, \quad \forall x, y \in \mathbb{R}^n.$$

Remark 2.13. Note that in any bounded domain U of $\mathbb{R}^n \setminus \{0\}$, $G(\cdot, y)$ satisfies all the properties of a weak solution of a uniformly elliptic equation. The fundamental solution of (2.4) also has the following properties (for more details, see [32, 35, 21]):

- $G(x, y) = G(y, x)$
- $G(\cdot, y) \in C^{1,\alpha}(U)$ for some $\alpha > 0$.
- The function $u(x) = \int_{\mathbb{R}^n} G(x, y) f(y) \, dy$ is a weak solution in $H^1_{\text{loc}}(\mathbb{R}^n)$ of the equation $-\mathcal{L}u = f$ for any $f \in C^\infty_0(\mathbb{R}^n)$.
- When the coefficients a_{ij} are constants, the fundamental solution can be given explicitly as

$$(2.6) \qquad G^0(x, y) := \frac{1}{(n-2)\alpha_n \sqrt{\det A}} \left(\sum_{ij} (A^{-1})_{ij}(x_i - y_i)(x_j - y_j) \right)^{\frac{2-n}{n}},$$

  where $(A^{-1})_{ij}$ are the elements of the inverse matrix of (a_{ij}), $\det A$ is the determinant of (a_{ij}) and α_n is the volume of the unit ball in $\mathbb{R}^n$.

Moreover, in a periodic setting, the results in [1, Proposition 5] gives the bounds on the gradient of the fundamental solution.

Lemma 2.14 (cf. [1, Proposition 5]). *If $n \geq 2$ and A is periodic then the fundamental solution G of (2.4) satisfies the following gradient estimates:*

$$(2.7) \qquad \exists C > 0, \qquad \forall x \in \mathbb{R}^n, \qquad \forall y \in \mathbb{R}^n, \qquad |D_x G(x, y)| \leq \frac{C}{|x - y|^{n-1}},$$

$$(2.8) \qquad \exists C > 0, \qquad \forall x \in \mathbb{R}^n, \qquad \forall y \in \mathbb{R}^n, \qquad |D_y G(x, y)| \leq \frac{C}{|x - y|^{n-1}}.$$

Using the technique of G-convergence, the authors in [47] established the results on the homogenization and the asymptotic behavior of the fundamental solution of (2.4). We refer to [47, 11] for the definition of G-convergence and more details of the homogenization problems.

Lemma 2.15 (cf. [47, Chapter III, Theorem 2]). *Let $n \geq 3$, A satisfy (1.3), (1.4) and G^ε be the fundamental solution of*

$$(2.9) \qquad -\mathcal{L}^\varepsilon u := -D_i \left(a_{ij} \left(\frac{x}{\varepsilon} \right) D_j u \right) = 0.$$

Then G^ε converges locally uniformly to G^0 in $\mathbb{R}^{2n} \setminus \{x = y\}$ as $\varepsilon \to 0$, where G^0 is the fundamental solution of

$$(2.10) \qquad -\mathcal{L}^0 u := -q_{ij} D_{ij} u = 0,$$

and (q_{ij}) is a constant symmetric positive definite matrix depending only on a_{ij}. Moreover, if we denote G as the fundamental solution of (2.4), then we will have the asymptotic expression

$$(2.11) \qquad G(x, y) = G_0(x, y) + |x - y|^{2-n}\theta(x, y),$$

where $\theta(x, y) \to 0$ as $|x - y| \to \infty$ uniformly on the set $\{|x| + |y| < a|x - y|\}$, a is any fixed positive constant.

2.3. **Rescaling.** Recall that we use the notations $\mathcal{L}, \mathcal{L}^\lambda$ for the operators as defined in (1.10). Following [39, 36, 38], for $\lambda > 0$ and $n \geq 3$ we rescale the solution v of the problem (1.1) as

$$v^\lambda(x, t) = \lambda^{\frac{n-2}{n}} v(\lambda^{\frac{1}{n}} x, \lambda t).$$

Clearly v^λ is a solution of

(2.12)
$$\begin{cases} \lambda^{\frac{2-n}{n}} v_t^\lambda - \mathcal{L}^\lambda v^\lambda = 0 & \text{in } \Omega(v^\lambda) \setminus K^\lambda, \\[2mm] v^\lambda = \lambda^{\frac{n-2}{n}} & \text{on } K^\lambda, \\[2mm] \dfrac{v_t^\lambda}{|Dv^\lambda|} = g^\lambda(x) a_{ij}^\lambda(x) D_j v^\lambda \nu_i & \text{on } \Gamma(v^\lambda), \\[2mm] v^\lambda(\cdot, 0) = v_0^\lambda & \text{in } \mathbb{R}^n, \end{cases}$$

where $K^\lambda := K/\lambda^{\frac{1}{n}}, \Omega_0^\lambda := \Omega_0/\lambda^{\frac{1}{n}}, g^\lambda(x) = g(\lambda^{\frac{1}{n}} x), a_{ij}^\lambda(x) = a_{ij}(\lambda^{\frac{1}{n}} x)$ and $v_0^\lambda(x) = \lambda^{\frac{n-2}{n}} v_0(\lambda^{\frac{1}{n}} x)$.

The corresponding rescaling of the weak solution u of the variational problem (2.1) can be shown to be (see [39, 36, 38])

$$u^\lambda(x, t) = \lambda^{-\frac{2}{n}} u(\lambda^{\frac{1}{n}} x, \lambda t),$$

which solves the rescaled obstacle problem

(2.13)
$$\begin{cases} u^\lambda(\cdot, t) \in \mathcal{K}^\lambda(t), & 0 < t < \infty, \\[2mm] (\lambda^{\frac{2-n}{n}} u_t^\lambda - \mathcal{L}^\lambda u^\lambda)(\varphi - u^\lambda) \geq f^\lambda(x)(\varphi - u^\lambda) & \text{a.e. } (x, t) \in \mathbb{R}^n \times (0, \infty) \\[2mm] & \text{for any } \varphi \in \mathcal{K}^\lambda(t), \\[2mm] u^\lambda(x, 0) = 0, \end{cases}$$

where $\mathcal{K}^\lambda(t) = \{\varphi \in H^1(\mathbb{R}^n), \varphi \geq 0, \varphi = \lambda^{\frac{n-2}{n}} t \text{ on } K^\lambda\}$ and $f^\lambda(x) = f(\lambda^{\frac{1}{n}} x)$.

Remark 2.16. The admissible set $\mathcal{K}^\lambda(t)$ can be defined in this way due to [38, Remark 2.13]. Note that for any fixed time t, the admissible set $\mathcal{K}^\lambda(t)$ depends on λ.

2.3.1. *Convergence of the rescaled fundamental solution.* By Lemma 2.15, we have the following convergence result on the rescaled fundamental solution.

Lemma 2.17. *Let G be the fundamental solution of* (2.4) *in dimension $n \geq 3$ and G^λ be its rescaling as*

$$G^\lambda(x, y) = \lambda^{\frac{n-2}{n}} G(\lambda^{\frac{1}{n}} x, \lambda^{\frac{1}{n}} y).$$

Then G^λ is the fundamental solution of

(2.14)
$$-\mathcal{L}^\lambda u = 0,$$

and $|G^\lambda(x, y) - G^0(x, y)| \to 0$ uniformly on every compact subset of $\mathbb{R}^{2n} \setminus \{(x, x) \in \mathbb{R}^{2n}\}$ where G^0 is the fundamental solution of (2.10).

Proof. We will show that G^λ is the fundamental solution of (2.14), then the result follows directly from Lemma 2.15 with $\varepsilon = \lambda^{-\frac{1}{n}}$.

For simplicity, we will check that G^λ satisfies the definition of the fundamental solution of (2.14) for fixed $y = 0, F(x) = G(x, 0)$ and $F^\lambda(x) := \lambda^{\frac{n-2}{n}} F(\lambda^{\frac{1}{n}} x)$.

Indeed, we have $D_j F^\lambda(x) = \lambda^{\frac{n-1}{n}} D_j F(\lambda^{\frac{1}{n}} x)$. Take a function $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$
\begin{aligned}
\int_{\mathbb{R}^n} a_{ij}^\lambda(x) D_j F^\lambda(x) D_i \varphi(x)\, dx &= \int_{\mathbb{R}^n} \lambda^{\frac{n-1}{n}} a_{ij}(\lambda^{\frac{1}{n}} x) D_j F(\lambda^{\frac{1}{n}} x) D_i \varphi(x)\, dx \\
&= \int_{\mathbb{R}^n} \lambda^{-\frac{1}{n}} a_{ij}(y) D_j F(y) D_i \varphi(\lambda^{-\frac{1}{n}} y)\, dy \\
&= \int_{\mathbb{R}^n} a_{ij}(y) D_j F(y) D_i \tilde{\varphi}(y)\, dy \\
&= \tilde{\varphi}(0) = \varphi(0),
\end{aligned}
$$

where $\tilde{\varphi}(y) = \varphi(\lambda^{-\frac{1}{n}} y)$. Moreover, F^λ satisfy the estimate (2.5) since F has this property. Hence, by definition, F^λ is the fundamental solution of (2.14). $\square$

Remark 2.18. The rate of this convergence as well as the rate of convergence for derivatives were also derived in [3].

2.4. **Construction of barriers from a fundamental solution.** The main goal of this section is to construct a subsolution and a supersolution of (1.1) from a fundamental solution of the elliptic equation (2.4) so that we can use them as barriers to track the behavior of the support of a solution of (1.1).

From now on, we will let $\mathcal{L}^0$ be the limit of the operators of $\mathcal{L}^\lambda$ as in Lemma 2.15 and consider the fundamental solutions of (2.4), (2.14) and (2.10) with a pole at the origin as

$$
F(x) := G(x,0), \quad F^\lambda(x) := G^\lambda(x,0) = \lambda^{\frac{n-2}{n}} F(\lambda^{\frac{1}{n}} x), \qquad F^0(x) := G^0(x,0).
$$

Note that F^0 is preserved under the rescaling by (2.6).

2.4.1. *Construction of a supersolution.* Define

$$
\theta(x,t) := [C_1 F(x) - C_2 t^{\frac{2-n}{n}}]_+,
$$

where $[s]_+ := \max(s, 0)$ denotes the positive part of s and C_1, C_2 are non-negative constants to be chosen later. It easily follows that in $\{\theta > 0\} \setminus \{x = 0\}$,

$$
\begin{aligned}
\theta_t(x,t) &= \frac{C_2(n-2)}{n} t^{\frac{2-2n}{n}} \geq 0, \\
D\theta &= C_1 DF, \\
\mathcal{L}\theta &= 0, \\
\theta_t - \mathcal{L}\theta &\geq 0.
\end{aligned}
$$

Due to the estimates (2.5) and (2.7), there exists a constant C such that

$$
\begin{aligned}
(2.15) \qquad & C^{-1}|x|^{2-n} \leq F(x) \leq C|x|^{2-n}, \\
& |DF(x)| \leq C|x|^{1-n}.
\end{aligned}
$$

Then for $(x,t) \in \partial\{\theta > 0\}$ we have

$$
C_2 t^{\frac{2-n}{n}} = C_1 F(x) \geq C_1 C^{-1} |x|^{2-n},
$$

which yields

$$
t^{\frac{1}{n}} \leq \left(\frac{C_1}{CC_2} \right)^{\frac{1}{2-n}} |x|.
$$

Thus on $\partial\{\theta > 0\}$,

$$\theta_t(x,t) = \frac{C_2(n-2)}{n} t^{\frac{2-2n}{n}} \geq \frac{n-2}{n} \left(\frac{C_1}{C}\right)^{\frac{2-2n}{2-n}} C_2^{\frac{n}{2-n}} |x|^{2-2n}.$$

Fix any $t_0 > 0$. We can choose C_1 large enough and C_2 small enough such that

$$\theta_t(x,t) \geq M\beta C_1^2 C^2 |x|^{2-2n} \geq M\beta |D\theta(x,t)|^2 \quad \text{on } \partial\{\theta > 0\},$$
$$\theta > 1 \text{ on } K \text{ and } \theta(x,t_0) > v(x,t_0),$$

where α, β are the elliptic constants from (1.3). By (1.5), $\theta_t \geq g\, a_{ij}\, D_j\theta D_i\theta$ on $\partial\{\theta > 0\}$ and by (1.2), θ is a supersolution of (1.1) in $\mathbb{R}^n \times [t_0, \infty)$.

2.4.2. *Construction of a subsolution.* Let h be the function constructed in [30, Appendix A] with $\mathcal{L}h = n$, $Dh(x) = (A(x))^{-1}x$ and let $c, \tilde{c} > 0$ be constants such that

(2.16) $$c|x|^2 \leq h(x) \leq \tilde{c}|x|^2.$$

Consider the function

(2.17) $$\theta(x,t) := \left[c_1 F(x) + \frac{c_2 h(x)}{t} - c_3 t^{\frac{2-n}{n}}\right]_+ \chi_E(x,t)$$

with non-negative constants c_1, c_2, c_3 to be chosen later, where

$$E := \left\{(x,t) : \frac{\partial F_b}{\partial r}(|x|,t) < 0, t > 0\right\}, \quad F_b(r,t) := Cc_1 r^{2-n} + \frac{c_2 \tilde{c} r^2}{t} - c_3 t^{\frac{2-n}{n}},$$

$C, \tilde{c}$ are constants as in (2.15), (2.16). We claim that we can choose constants c_1, c_2, c_3, t_0 such that θ is a subsolution of (1.1) for $t \in [t_0, \infty)$. The differentiation of θ on the set $\{\theta > 0\} \setminus \{x = 0\}$ yields

$$D\theta(x,t) = c_1 DF(x) + \frac{c_2 A(x)^{-1}x}{t},$$

(2.18) $$\mathcal{L}\theta(x,t) = \frac{c_2 n}{t},$$

$$\theta_t(x,t) = -\frac{c_2 h(x)}{t^2} + \frac{c_3(n-2)}{n} t^{\frac{2-2n}{n}} = t^{\frac{2-2n}{n}} \left[\frac{c_3(n-2)}{n} - \frac{c_2 h(x)}{t^{\frac{2}{n}}}\right],$$

and thus

$$\theta_t(x,t) - \mathcal{L}\theta(x,t) = t^{\frac{2-2n}{n}} \left[\frac{c_3(n-2)}{n} - \frac{c_2 h(x)}{t^{\frac{2}{n}}} - \frac{c_2 n}{t^{\frac{2-n}{n}}}\right] < 0 \quad \text{for } t \gg 1.$$

Thus, we can choose t_0 large enough such that $\theta_t - \mathcal{L}\theta < 0$ for $t \geq t_0$.

Now we will prove the continuity of θ. We have

(2.19) $$0 \leq \theta(x,t) \leq [F_b(|x|,t)]_+ \chi_E(x,t) =: F_b^+(x,t),$$

and hence $\Omega_t(\theta) \subset \Omega_t(F_b^+)$ for all t. We see that

(2.20) $$\frac{\partial F_b}{\partial r}(r,t) = Cc_1(2-n)r^{1-n} + \frac{2c_2\tilde{c}r}{t} < 0 \Leftrightarrow r < \left(\frac{Cc_1(n-2)}{2c_2\tilde{c}}\right)^{\frac{1}{n}} t^{\frac{1}{n}} =: r_0(t)$$

and hence $E = \{(x,t) : |x| < r_0(t), t > 0\}$. Clearly θ is continuous in the set $\{\theta > 0\} \setminus \{x = 0\}$. Furthermore, θ is continuous in $E \setminus \{x = 0\}$ and $\theta = 0$ on E^c. We will show that we can choose the constants such that θ is continuous through boundary of E. Indeed, for $(x_0, t) \in \partial E$, $t > 0$,

$$F_b(|x_0|,t) = F_b(r_0(t),t) = C_{F_b} t^{\frac{2-n}{n}},$$

where $C_{F_b} = (Cc_1)^{\frac{2}{n}}(c_2\tilde{c})^{\frac{n-2}{n}}\left[\left(\frac{n-2}{2}\right)^{\frac{2}{n}}\frac{n}{n-2}\right] - c_3$. We can choose c_1, c_2, c_3 such that $C_{F_b} < 0$. Then $F_b(|x_0|, t) < 0$ for all $(x_0, t) \in \partial E$, $t > 0$. Since $(x, t) \mapsto F_b(|x|, t)$ is continuous in a neighborhood of ∂E, we deduce by (2.19) that $\theta = 0$ in a neighborhood of ∂E and therefore it is continuous across ∂E. Note that $C_{F_b} < 0$ if and only if

$$(2.21) \qquad c_3 \geq C_0(c_1)^{\frac{2}{n}}(c_2)^{\frac{n-2}{n}},$$

where C_0 is a constant depending only on $n, C, \tilde{c}$.

We finally need to show that we can choose suitable constants such that θ satisfies the subsolution condition on the free boundary.

We first note that $\theta(x, t) \geq \tilde{\theta}(x, t) := \left[Cc_1|x|^{2-n} - c_3t^{\frac{2-n}{n}}\right]_+$. Then $\Omega(\tilde{\theta}) \subset \Omega(\theta)$, or more precisely, there exists a constant $\tilde{C}$ such that

$$(2.22) \qquad |x| \geq \tilde{C}t^{\frac{1}{n}} \text{ for all } (x, t) \in \partial\{\theta > 0\}.$$

By (2.18) we have

$$\theta_t(x, t) \leq c_3 t^{\frac{2-2n}{n}},$$

$$|D\theta(x, t)|^2 = c_1^2|DF(x)|^2 + \frac{2c_1c_2}{t}DF(x) \cdot A^{-1}x + \frac{c_2^2}{t^2}|A^{-1}x|^2,$$

$$\geq \frac{2c_1c_2}{t}DF(x) \cdot A^{-1}x + \frac{c_2^2}{t^2}|A^{-1}x|^2.$$

Since A is a symmetric bounded matrix satisfying the ellipticity (1.3), then these properties also hold for A^{-1} and A^{-2} with appropriate constants. Hence, for $(x, t) \in \partial\{\theta > 0\}$,

$$|D\theta(x, t)|^2 \geq \frac{c_2^2}{t^2}\tilde{\alpha}|x|^2 - \frac{2c_1c_2}{t}C_A|DF(x)||x| \qquad \text{for some } \tilde{\alpha}, C_A > 0$$

$$\geq \frac{c_2^2}{t^2}\tilde{\alpha}|x|^2 - \frac{2c_1c_2}{t}CC_A|x|^{2-n} \qquad \text{(by (2.15))}$$

$$\geq \left(c_2^2\tilde{\alpha}\tilde{C}^2 - 2c_1c_2CC_A\tilde{C}^{2-n}\right)t^{\frac{2-2n}{n}} \qquad \text{(by (2.22))}.$$

We want to choose c_1, c_2, c_3 such that $\theta_t \leq m\alpha|D\theta|^2$ on $\partial\{\theta > 0\}$, which will hold if

$$(2.23) \qquad c_3 \leq m\alpha\left(c_2^2\tilde{\alpha}\tilde{C}^2 - 2c_1c_2CC_A\tilde{C}^{2-n}\right) =: C_0^1c_2^2 - C_0^2c_1c_2,$$

where C_0^1, C_0^2 are fixed positive constants. Then by (2.9), $\theta_t \leq g\, a_{ij}\, D_j\theta D_i\theta$ on $\partial\{\theta > 0\}$.

The conditions (2.21) and (2.23) hold if we choose some suitable c_1, c_2, c_3. For example, fix any $c_1 > 0$, choose c_2 large enough such that

$$C_0(c_1)^{\frac{2}{n}}(c_2)^{\frac{n-2}{n}} < C_0^1c_2^2 - C_0^2c_1c_2.$$

Note that the above inequality holds for c_2 large enough since for a fixed $c_1 > 0$, the right hand side tends to ∞ as $c_2 \to \infty$ faster than the left hand side. Then (2.21) and (2.23) hold for any c_3 which is between these two numbers. Fix t_0 such that $\theta_t - \mathcal{L}\theta < 0$ in $\{\theta > 0\}$ for chosen c_2, c_3 and $t \geq t_0$. Choosing a smaller c_1 if it is needed, we can assume that the support of $\theta(\cdot, t_0)$ is contained in $\Omega_{t_0}(v), \theta(x, t_0) \leq v(x, t_0)$ and $\theta < 1$ on ∂K. Thus, with the help of (1.2), we see that θ is a subsolution of the Stefan problem (1.1) for that choice of constants.

2.4.3. *Some results on the barriers for the Stefan problem* (1.1). Due to the construction above, we can use the functions of the form

$$(2.24) \qquad \theta(x,t) := [C_1 F(x) - C_2 t^{\frac{2-n}{n}}]_+$$

with $C_1, C_2 > 0$ as barriers for the Stefan problem (1.1). Since our purpose is to study the asymptotic behavior, we first observe the convergence of the rescaled barriers.

Lemma 2.19. *Let θ be a function of the form* (2.24) *and* $\theta^\lambda := \lambda^{\frac{n-2}{n}} \theta(\lambda^{\frac{1}{n}} x, \lambda t)$. *Then* $\theta^\lambda \to \theta^0$ *locally uniformly in* $(\mathbb{R}^n \setminus \{0\}) \times [0, \infty)$, *where*

$$(2.25) \qquad \theta^0(x,t) := [C_1 F^0(x) - C_2 t^{\frac{2-n}{n}}]_+.$$

Proof. We have

$$\theta^\lambda(x,t) = [C_1 F^\lambda(x) - C_2 t^{\frac{2-n}{n}}]_+,$$

where $F^\lambda(x) = \lambda^{\frac{n-2}{n}} F(\lambda^{\frac{1}{n}} x)$. By Lemma 2.17, $F^\lambda \to F^0$ locally uniformly in $\mathbb{R}^n \setminus \{0\}$ and the lemma follows. $\qquad \square$

Moreover we will also need to know the integral of the barriers in time to analyze the weak solution of the Stefan problem (1.1).

Lemma 2.20. *Let $\Theta(x,t) := \int_0^t \theta(x,s) \, ds$. Then $\Theta(x,t)$ has the form*

$$(2.26) \qquad \Theta(x,t) = C_1 F(x) t - \frac{C_2 n}{2} t^{\frac{2}{n}} + o(F(x)), \ as \ |x| \to 0.$$

Proof. We can derive (2.26) simply by integrating the function θ of the form (2.24). Since θ has the form (2.24), we see that

$$\begin{aligned} \theta > 0 \quad &\text{if } t > s(x), \\ \theta = 0 \quad &\text{if } t \le s(x), \end{aligned} \quad \text{where } s(x) = \left(\frac{C_1}{C_2} F(x) \right)^{\frac{n}{2-n}}.$$

Thus,

$$\Theta(x,t) = \begin{cases} 0, & t \le s(x), \\ \displaystyle\int_{s(x)}^t (C_1 F(x) - C_2 s^{\frac{2-n}{n}}) \, ds, & t > s(x). \end{cases}$$

When $t > s(x)$,

$$\begin{aligned} \Theta(x,t) &= C_1 F(x) t - \frac{C_2 n}{2} t^{\frac{2}{n}} - C_1 F(x) s(x) + \frac{C_2 n}{2} (s(x))^{\frac{2}{n}} \\ &= C_1 F(x) t - \frac{C_2 n}{2} t^{\frac{2}{n}} + \frac{n-2}{2} \frac{(C_1)^{\frac{2}{2-n}}}{(C_2)^{\frac{n}{2-n}}} (F(x))^{\frac{2}{2-n}} \\ &= C_1 F(x) t - \frac{C_2 n}{2} t^{\frac{2}{n}} + C(F(x))^{\frac{2}{2-n}}. \end{aligned}$$

Since $F(x)$ has a singularity at $x = 0$ by (2.5) then $s(x) \to 0$ and $C(F(x))^{\frac{2}{2-n}} = o(F(x))$ as $|x| \to 0$, which completes the proof. $\qquad \square$

From these barriers, we can obtain the rate of expansion of the support for viscosity solutions.

Lemma 2.21. *Let $n \geq 3$ and v be a viscosity solution of (1.1). There exists $t_0 > 0$ and constants $C, C_1, C_2 > 0$ such that for $t \geq t_0$,*

$$C_1 t^{\frac{1}{n}} \leq \min_{\Gamma_t(v)} |x| \leq \max_{\Gamma_t(v)} |x| \leq C_2 t^{\frac{1}{n}}$$

and for $0 \leq t \leq t_0$,

$$\max_{\Gamma_t(v)} |x| \leq C_2.$$

Moreover,

$$0 \leq v(x,t) \leq C|x|^{2-n}.$$

Proof. We deduce the bound for $v(x,t)$ first. Let $F(x)$ be the fundamental solution of the elliptic equation (2.4) as in Section 2.4. Then $\hat{\theta} = CF(x)$ is a stationary solution of the equation $v_t - \mathcal{L}v = 0$. Its integral in time is also a solution of the variational inequality problem with $\hat{f} = CF(x)$. If we take C large enough then $\hat{f} \geq f$ and $\hat{\theta} \geq 1$ on K. Applying the comparison principle for the variational problem, [39, Proposition 2.2], we have $v(x,t) \leq CF(x) \leq \tilde{C}|x|^{2-n}$ by (2.5).

The bound on the support of $v(\cdot, t)$ at all times has been proved in [30, Lemma 3.6]. ∎

Now consider θ_1, θ_2 that are respectively a subsolution and a supersolution of the Stefan problem (1.1) for $t \geq t_0$ as constructed in Section 2.4.1 and 2.4.2. The bounds on the support of v for $t \geq t_0$ follow directly from the behavior of the supports of θ_1, θ_2. □

2.5. Limit problems.
The expected limit problem is the corresponding Hele-Shaw type problem with a point source.

2.5.1. *Limit problem for v^λ.* We expect v^λ to converge to a solution of

$$(2.27) \qquad \begin{cases} \mathcal{L}^0 v = 0 & \text{in } \{v > 0\}, \\[2mm] \dfrac{v_t}{|Dv|} = \dfrac{1}{L} q_{ij} D_j v \nu_i & \text{on } \partial\{v > 0\}, \\[2mm] \lim_{|x| \to 0} \dfrac{v}{F^0} = C, \\[2mm] v(x,0) = 0 & \text{in } \mathbb{R}^n \setminus \{0\}, \end{cases}$$

where C, L are positive constants, q_{ij} are constants of the operator $\mathcal{L}^0$ and F^0 is the fundamental solution of (2.10).

Since $Q := (q_{ij})$ is symmetric and positive definite, we can write $Q = P^2$, where P is also a symmetric positive definite matrix. Let $\tilde{v}(x,t) := v(Px,t)$. A direct computation then shows that the problem (2.27) becomes the classical Hele-Shaw problem with a point source for function $\tilde{v}$,

$$(2.28) \qquad \begin{cases} \Delta \tilde{v} = 0 & \text{in } \{\tilde{v} > 0\}, \\[2mm] \tilde{v}_t = \dfrac{1}{L} |D\tilde{v}|^2 & \text{on } \partial\{v > 0\}, \\[2mm] \lim_{|x| \to 0} \dfrac{\tilde{v}}{|x|^{2-n}} = C, \\[2mm] \tilde{v}(x,0) = 0 & \text{in } \mathbb{R}^n \setminus \{0\}. \end{cases}$$

The problem (2.28) has a unique classical solution $\tilde{V}$ which is given explicitly (see [36, 38], for instance). Thus (2.27) has unique classical solution $V(x,t) := \tilde{V}(P^{-1}x, t)$, which is continuous in $(\mathbb{R}^n \setminus \{0\}) \times [0, \infty)$.

2.5.2. *Limit problem for* u^λ. Suppose that $V = V_{C,L}$ is the classical solution of (2.27) above and set

$$(2.29) \qquad U(x,t) := \int_0^t V(x,s)\,ds.$$

It is known that the time integral of the solution of the classical Hele-Shaw problem with a point source (2.28) satisfies an obstacle problem derived in [36]. Following [36] and using a change of variables again, we see that U uniquely solves the following problem, which is our limit variational problem:

$$(2.30) \qquad \begin{cases} w \in \mathcal{K}_t, \\ q(w,\phi) \geq \langle -L, \phi \rangle, & \forall \phi \in W_1, \\ q(w,\psi w) = \langle -L, \psi w \rangle, & \forall \psi \in W_2, \end{cases}$$

where

$$\mathcal{K}_t = \left\{ \varphi \in \bigcap_{\varepsilon > 0} H^1(\mathbb{R}^n \setminus B_\varepsilon) \cap C(\mathbb{R}^n \setminus B_\varepsilon) : \varphi \geq 0, \lim_{|x| \to 0} \frac{\varphi(x)}{F^0(x)} = Ct \right\},$$

$$(2.31) \qquad W_1 = \left\{ \phi \in H^1(\mathbb{R}^n \setminus B_\varepsilon) : \phi \geq 0, \phi = 0 \text{ on } B_\varepsilon \text{ for some } \varepsilon > 0 \right\},$$

$$(2.32) \qquad W_2 = W_1 \cap C^1(\mathbb{R}^n).$$

2.5.3. *Near-field limit.* Using the boundedness provided by Lemma 2.21, we have the following general near-field limit result similar to [39].

Theorem 2.22 (Near-field limit). *The viscosity solution v of the Stefan problem (1.1) converges to the unique solution $P = P(x)$ of the exterior Dirichlet problem*

$$(2.33) \qquad \begin{cases} \mathcal{L}P = 0, & x \in \mathbb{R}^n \setminus K, \\ P = 1, & x \in K, \\ \lim_{|x| \to \infty} P(x) = 0, \end{cases}$$

as $t \to \infty$ uniformly on compact subsets of $\overline{K^c}$.

Proof. Follow the arguments in proof of [39, Lemma 8.4] and note that by Lemma 2.21 the support of v expands to the whole space as time $t \to \infty$. $\square$

The results on the isolated singularity of solutions of linear elliptic equations in [46] allow us to deduce the asymptotic behavior of P as $|x| \to \infty$.

Lemma 2.23. *There exists a constant $C_* = C_*(K,n)$ such that the solution P of the problem (2.33) satisfies*

$$\lim_{|x| \to \infty} \frac{P(x)}{F(x)} = C_*,$$

where $F(x)$ is the fundamental solution of the elliptic equation $-\mathcal{L}v = 0$ in $\mathbb{R}^n$.

Proof. Lemma 2.23 is a direct corollary of [46, Theorem 5]. The arguments follow the same techniques as in [39, Lemma 4.3] using a general Kelvin transform and Green's function for linear elliptic equations. Following [39, Lemma 4.3], it can also be shown that the constant C_* depends continuously on the data of the fixed boundary $\Gamma = \partial K$. $\square$

3. Uniform convergence of the rescaled variational solutions

The purpose of this section is to show the first main result on the uniform convergence of the rescaled variational solutions, which is similar to [38, Theorem 3.1].

Theorem 3.1. *Let u be the unique solution of variational problem (2.1) and u^λ be its rescaling. Let $U_{A,L}$ be the unique solution of limit problem (2.30) where $A = C_*$ as in Lemma 2.23, and $L = \left\langle \frac{1}{g} \right\rangle$ as in Lemma 3.2. Then the functions u^λ converge locally uniformly to $U_{A,L}$ as $\lambda \to \infty$ on $(\mathbb{R}^n \setminus \{0\}) \times [0, \infty)$.*

The classical homogenization results of variational inequalities are usually stated for a fixed bounded domain. Since our admissible set $\mathcal{K}^\lambda(t)$ defined in Section 2.3 changes with λ, we will need to refine the proof. We will use the techniques of the Γ-convergence introduced in [11] and [30]. Note that these techniques can be applied not only for the periodic case but also for stationary ergodic coefficients over a probability space $(A, \mathcal{F}, P)$.

3.1. **The averaging property of media and the Γ-convergence.** We recall the following lemma on the averaging property of periodic media, which also holds for more general stationary ergodic media.

Lemma 3.2 (cf. [29, Section 4, Lemma 7], see also [36]). *For a given g satisfying (1.4), there exists a constant, denoted by $\left\langle \frac{1}{g} \right\rangle$, such that if $\Omega \subset \mathbb{R}^n$ is a bounded measurable set and if $\{u^\varepsilon\}_{\varepsilon > 0} \subset L^2(\Omega)$ is a family of functions such that $u^\varepsilon \to u$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$, then*

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{1}{g\left(\frac{x}{\varepsilon}\right)} u^\varepsilon(x) \, dx = \int_\Omega \left\langle \frac{1}{g} \right\rangle u(x) \, dx.$$

The quantity $\left\langle \frac{1}{g} \right\rangle$ in periodic setting is the average of $\frac{1}{g}$ over one period.

We also need some basic concepts and results of the Γ-convergence which are taken from [11]. Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the functional

$$(3.1) \qquad J^\lambda(u, \Omega) := \begin{cases} \displaystyle\int_\Omega a_{ij}(\lambda^{\frac{1}{n}} x) D_i u D_j u \, dx & \text{if } u \in H^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Definition 3.3 (cf. [11, Proposition 8.1]). Let X be a metric space. A sequence of functionals F_h is said to $\Gamma(X)$-converge to F if the following conditions are satisfied:

(i) For every $u \in X$ and for every sequence (u_h) converging to u in X, we have

$$F(u) \le \liminf_{h \to 0} F_h(u_h).$$

(ii) For every $u \in X$, there exists a sequence (u_h) converging to u in X, such that

$$F(u) = \lim_{h \to 0} F_h(u_h).$$

It is known that the $\Gamma(L^2)$-convergence of J^λ is equivalent to the G-convergence of elliptic operator $\mathcal{L}^\lambda$ (see [11, Theorem 22.4] and [30, Theorem 4.3]) and we have a crucial result on Gamma-convergence of J^λ as follows.

Theorem 3.4 (cf. [30, Theorem 4.3]). *The functionals J^λ $\Gamma(L^2)$-converge as $\lambda \to \infty$ to a functional J^0, where J^0 is a quadratic functional of the form*

$$J^0(u) := \begin{cases} \displaystyle\int_\Omega q_{ij} D_i u D_j u \, dx & \text{if } u \in H^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Here the constants q_{ij} are the coefficients of the limit operator $\mathcal{L}^0$ as in Lemma 2.15.

To deal with the Dirichlet boundary condition, we need to use cut-off functions and the *fundamental estimate* below. Here we denote as $\mathcal{A}$ the class of all open subsets of Ω.

Definition 3.5. [11, Definition 18.1] Let $A', A'' \in \mathcal{A}$ with $A' \Subset A''$. We say that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-off function* between A' and A'' if $\varphi \in C_0^\infty(A''), 0 \leq \varphi \leq 1$ on $\mathbb{R}^n$, and $\varphi = 1$ in a neighborhood of $\overline{A'}$.

Definition 3.6. [11, Definition 18.2] Let $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be a non-negative functional. We say that F satisfies the *fundamental estimate* if for every $\varepsilon > 0$ and for every $A', A'', B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ with the following property: for very $u, v \in L^p(\Omega)$, there exists a cut-off function φ between A' and A'', such that

$$\begin{aligned}
(3.2) \qquad F(\varphi u + (1-\varphi)v, A' \cup B) &\leq (1+\varepsilon)(F(u, A'') + F(v, B)) \\
&\quad + \varepsilon(\|u\|_{L^p(S)}^p + \|v\|_{L^p(S)}^p + 1) + M\|u - v\|_{L^p(S)}^p,
\end{aligned}$$

where $S = (A'' \setminus A') \cap B$. Moreover, if $\mathcal{F}$ is a class of non-negative functionals on $L^p(\Omega) \times \mathcal{A}$, we say that the fundamental estimate holds uniformly in $\mathcal{F}$ if each element F of $\mathcal{F}$ satisfies the fundamental estimate with M depending only on ε, A', A'', B, while φ may depend also on F, u, v.

The result in [11, Theorem 19.1] provides a wide class of integral functionals uniformly satisfying the fundamental estimate. In particular, the fundamental estimate holds uniformly in the class of all functionals of the form (3.1). Thus for every J^λ, there exists a cut-off function φ such that (3.2) hold with $F = J^\lambda$ and a constant M independent of λ.

3.2. Uniform convergence of rescaled variational solutions. Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. For a fixed $T > 0$, we can bound $\Omega_t(u^\lambda)$ by $B(0, R)$ for some $R > 0$, for all $0 \leq t \leq T$ and $\lambda > 0$ by Lemma 2.21. We will show the convergence in $Q_\varepsilon := \left(B(0, R) \setminus \overline{B(0, \varepsilon)} \right) \times [0, T]$ for some $\varepsilon > 0$.

We argue the same way as in the proof of [38, Theorem 3.2]. Using the uniform bound on u^λ, u_t^λ from Lemma 2.21 and the standard regularity estimates for an elliptic obstacle problem which hold uniformly in λ, we obtain a uniform Hölder estimate for u^λ. Then by the Arzelà-Ascoli theorem and a diagonalization argument, we can find a function $\bar{u} \in C((\mathbb{R}^n \setminus \{0\}) \times [0, \infty))$ and a subsequence $\{u^{\lambda_k}\} \subset \{u^\lambda\}$ such that

$$u^{\lambda_k} \to \bar{u} \text{ locally uniformly on } (\mathbb{R}^n \setminus \{0\}) \times [0, \infty) \text{ as } k \to \infty,$$

$$u^{\lambda_k}(\cdot, t) \to \bar{u}(\cdot, t) \text{ strongly in } H^1(\Omega_\varepsilon) \text{ for all } t \geq 0, \varepsilon > 0.$$

In the rest of the proof we show that the function $\bar{u}$ solves the limit problem (2.30), whose uniqueness then implies the convergence of the full sequence. We start with quantifying the singularity at the origin.

Lemma 3.7. *We have*

$$\lim_{|x|\to 0} \frac{\overline{u}(x,t)}{U_{C_*,L}(x,t)} = 1.$$

Proof. Let C_* as in Lemma 2.23 and F be the fundamental solution of (2.4) as in Section 2.4. Fix $\varepsilon > 0$. By Lemma 2.23, there exists a large enough such that

$$(3.3) \qquad\qquad \left|\frac{P(x)}{F(x)} - C_*\right| < \frac{\varepsilon}{2}, \quad \text{in } \{|x| \geq a\}$$

and $K \subset \{|x| < a\}$. In particular, (3.3) holds for every $x, |x| = a$.

The set $\{|x| = a\}$ is a compact subset of $\mathbb{R}^n \setminus K$. Then by Theorem 2.22, there exists $t_0 > 0$ such that for all $t \geq t_0$,

$$\left|\frac{v(x,t)}{F(x)} - \frac{P(x)}{F(x)}\right| < \frac{\varepsilon}{2}, \quad \text{for all } x, |x| = a.$$

By triangle inequality we have for all $t \geq t_0$, for all x such that $|x| = a$,

$$\left|\frac{v(x,t)}{F(x)} - C_*\right| < \varepsilon.$$

Let $\Phi(x,t)$ be the fundamental solution of the parabolic equation

$$(3.4) \qquad\qquad\qquad\qquad u_t - \mathcal{L}u = 0.$$

As shown in [16, 2], such unique fundamental solution exists and satisfies

$$(3.5) \qquad\qquad N^{-1}t^{-\frac{n}{2}}e^{-\frac{N|x|^2}{t}} \leq \Phi(x,t) \leq Nt^{-\frac{n}{2}}e^{-\frac{|x|^2}{Nt}}$$

for some $N > 0$. We consider θ_1, θ_2 as follows:

$$\theta_1(x,t) := \left[(C_* - \varepsilon)F(x) + \frac{c_2 h(x)}{t} - c_3 t^{\frac{2-n}{n}}\right]_+ \chi_E(x,t),$$

$$\theta_2(x,t) := (C_* + \varepsilon)F(x) + C_2\Phi(x,t),$$

where E, $h(x)$ were defined as in Section 2.4.2. We will show that we can choose the coefficients such that θ_1 is a subsolution and θ_2 is a supersolution of (1.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$ for some t_0. Since we fix the first coefficient of θ_1 and θ_2, we need to check the initial conditions carefully.

Note that on the set $\{|x| = a\}$, $\theta_1 \to (C_* - \varepsilon)F(x)$ and $\theta_2 \to (C_* + \varepsilon)F(x)$ uniformly as $t \to \infty$. Thus we can choose a large time t_0 such that $\theta_1 \leq v \leq \theta_2$ on $\{|x = a|\} \times \{t \geq t_0\}$. By (2.20), we can choose c_2 large enough such that $\operatorname{supp}\theta_1(\cdot, t_0) \subset \{x : (x, t_0) \in E\} \subset B_a(0)$ and then $\theta_1(\cdot, t_0) \leq v(\cdot, t_0)$ in $\{|x| \geq a\}$. Following Section 2.4.2, by choosing larger c_2, t_0 if necessary and c_3 satisfying (2.21), (2.23), θ_1 is a subsolution of (1.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$.

Fix the time t_0 such that θ_1 is a subsolution of (1.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$ as above. By (2.5) and (3.5), $\theta_2 > 0$ in $\mathbb{R}^n$. Moreover, since $F(x)$ and $\Phi(x,t)$ are the fundamental solutions of (2.4) and (3.4) respectively, clearly $(\theta_2)_t - \mathcal{L}\theta_2 = 0$ in $\mathbb{R}^n \setminus \{0\}$. If we choose C_2 large enough then $\theta_2(\cdot, t_0) > v(\cdot, t_0)$ and θ_2 is a super solution of (1.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$.

By comparison principle, $\theta_1 \leq v \leq \theta_2$ in $\{|x| \geq a\} \times \{t \geq t_0\}$. Moreover, since $h(x) > 0$ then

$$\theta_1(x,t) \geq \tilde{\theta}_1(x,t) := \left[(C_* - \varepsilon)F(x) - c_3 t^{\frac{2-n}{n}} \right]_+.$$

Therefore $\tilde{\theta}_1^\lambda \leq v^\lambda \leq \theta_2^\lambda$ for λ is large enough.

Noting that $\Phi^\lambda(x,t) := \lambda^{\frac{n-2}{n}} \Phi(\lambda^{\frac{1}{n}} x, \lambda t) \to 0$ uniformly as $\lambda \to \infty$ by (3.5), then by Lemma 2.19, $\tilde{\theta}_1^\lambda, \theta_2^\lambda$ converge locally uniformly to θ_1^0, θ_2^0 of the form

$$\theta_1^0(x,t) := \left[(C_* - \varepsilon)F^0(x) - c_3 t^{\frac{2-n}{n}} \right]_+,$$

$$\theta_2^0(x,t) := (C_* + \varepsilon)F^0(x),$$

where F^0 is the fundamental solution of $-\mathcal{L}^0 u = 0$, $\mathcal{L}^0$ is the limit of the operators $\mathcal{L}^\lambda$ as in Lemma 2.15. Applying the same method as in [36] we have

$$(3.6) \qquad \int_0^t \theta_1^0(x,s)\,ds \leq \overline{u}(x,t) \leq \int_0^t \theta_2^0(x,s)\,ds.$$

By Lemma 2.20 we obtain

$$(C_* - \varepsilon)F^0(x)t - \frac{c_3 n}{2} t^{\frac{2}{n}} + o(F^0(x)) \leq \overline{u}(x,t) \leq (C_* + \varepsilon)F^0(x)t$$

as $|x| \to 0$. Dividing both sides of by $F^0(x)$ and taking the limit as $|x| \to 0$ we get

$$(C_* - \varepsilon)t \leq \liminf_{|x| \to 0} \frac{\overline{u}(x,t)}{F^0(x)} \leq \limsup_{|x| \to 0} \frac{\overline{u}(x,t)}{F^0(x)} \leq (C_* + \varepsilon)t.$$

Since $\varepsilon > 0$ is arbitrary, we have the correct singularity by sending ε to 0. $\qquad \square$

Finally, we check that the limit function $\bar{u}$ satisfies the inequality and equality in (2.30).

Lemma 3.8. *For each $0 \leq t \leq T$, $\overline{w} = \overline{u}(\cdot, t)$ satisfies*

$$(3.7) \qquad q(\overline{w}, \phi) \geq \langle -L, \phi \rangle, \qquad\qquad \forall \phi \in W_1,$$

$$(3.8) \qquad q(\overline{w}, \psi\overline{w}) = \langle -L, \psi\overline{w} \rangle, \qquad\qquad \forall \psi \in W_2,$$

where $L = \left\langle \frac{1}{g} \right\rangle$ and W_1, W_2 were defined as in Section 2.5.2.

Proof. Fix $t \in [0, T]$ and take any $\phi \in W_1$. By continuity, we can choose ϕ with a compact support contained in $\Omega := B(0, R) \setminus \overline{B(0, \varepsilon_0)}$ for some $0 < \varepsilon_0 < R$. Let $w^k(x) := u^{\lambda_k}(x, t)$ and $\overline{\varphi} := \overline{w} + \phi \in H^1(\mathbb{R}^n)$. By Theorem 3.4, there exists a sequence $\{\varphi^k\}$ that converges strongly in $L^2(\Omega)$ to $\overline{\varphi}$ such that

$$(3.9) \qquad J^{\lambda_k}(\varphi^k, \Omega) \to J^0(\overline{\varphi}, \Omega).$$

We will show that we can modify φ^k into $\tilde{\varphi}^k$ such that $\tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ and all the convergences are preserved.

First, we see that $J^0(\overline{\varphi}, \Omega) < \infty$ since $\overline{\varphi} \in H^1(\Omega)$. By (3.9), $J^{\lambda_k}(\varphi^k, \Omega) < \infty$ and hence $\varphi^k \in H^1(\Omega)$ when k is large enough.

Next, we need to modify φ^k so that the boundary condition on K^{λ_k} is satisfied. Since $\overline{\varphi} \in H^1(\Omega)$, for every $\varepsilon > 0$, there exists a compact set $A(\varepsilon) \subset \Omega$ such that $\text{supp}\,\phi \subset A(\varepsilon)$ and

$$(3.10) \qquad \int_{\Omega \setminus A(\varepsilon)} |D\overline{\varphi}|^2\,dx < \varepsilon.$$

Let $A'(\varepsilon), A''(\varepsilon)$ such that $A(\varepsilon) \subset A'(\varepsilon) \Subset A''(\varepsilon) \Subset \Omega$ and $B(\varepsilon) = \Omega \setminus A(\varepsilon)$. By [11, Theorem 19.1], the fundamental estimate (3.2) holds uniformly in the class of all functionals of the form (3.1). Thus there exists a constant $M \geq 0$ independent of λ_k and a sequence of cut-off functions $\xi_\varepsilon^k \in C_0^\infty(A''(\varepsilon)), 0 \leq \xi_\varepsilon^k \leq 1, \xi_\varepsilon^k = 1$ in a neighborhood of $\overline{A'(\varepsilon)}$ such that

(3.11)
$$
\begin{aligned}
J^{\lambda_k}(\xi_\varepsilon^k \varphi^k + (1 - \xi_\varepsilon^k)(w^k + \phi), \Omega) \leq &(1 + \varepsilon)(J^{\lambda_k}(\varphi^k, A''(\varepsilon)) + J^{\lambda_k}(w^k + \phi, B(\varepsilon))) \\
& + \varepsilon(\|\varphi^k\|_{L^2(\Omega)}^2 + \|w^k + \phi\|_{L^2(\Omega)}^2 + 1) \\
& + M\|\varphi^k - w^k - \phi\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Define

$$
\varphi_\varepsilon^k(x) := \begin{cases} \xi_\varepsilon^k(x)\varphi^k(x) + (1 - \xi_\varepsilon^k(x))(w^k(x) + \phi(x)) & \text{if } x \in \Omega, \\ w^k(x) & \text{if } x \notin \Omega. \end{cases}
$$

Then $\varphi_\varepsilon^k \in H^1(\mathbb{R}^n), \|\varphi_\varepsilon^k - \bar\varphi\|_{L^2(\Omega)} \leq \|\varphi^k - \bar\varphi\|_{L^2(\Omega)} + \|w^k + \phi - \bar\varphi\|_{L^2(\Omega)} \to 0$ as $k \to \infty$ and $\varphi_\varepsilon^k - w^k$ has compact support in Ω.

By ellipticity (1.3) we have

(3.12)
$$
J^{\lambda_k}(w^k + \phi, B(\varepsilon)) \leq \beta \int_{B(\varepsilon)} |D(w^k + \phi)|^2 \, dx.
$$

In view of (3.10), choose the sequence $\varepsilon_n := \frac{1}{n}$ and denote $\varphi_n^k := \varphi_{\varepsilon_n}^k$. By (3.11), (3.12), and the convergences $\varphi_n^k \to \overline{\varphi}$ in $L^2(\Omega)$ and $w^k \to \overline{w}$ in $H^1(\Omega)$ as $k \to \infty$, for each n there exists $k_0(n)$ such that

(3.13)
$$
\begin{cases} \|\varphi_n^k - \overline\varphi\|_{L^2(\Omega)} \leq \min\left\{\dfrac{1}{n}, \dfrac{1}{Mn}\right\}, \\[2mm] J^{\lambda_k}(\varphi_n^k, \Omega) \leq \left(1 + \dfrac{1}{n}\right)\left(J^0(\overline\varphi, \Omega) + \dfrac{\beta + 1}{n}\right) + \dfrac{1}{n}\left(2\|\overline\varphi\|_{L^2(\Omega)} + \dfrac{1}{n} + 1\right) + \dfrac{2}{n}, \end{cases}
$$

for every $k \geq k_0(n)$. We can choose $k_0(n)$ such that k_0 is an increasing function of n and $k_0(n) \to \infty$ as $n \to \infty$. We will form a new sequence $\{\hat\varphi^k\}$ from the class of sequences $\{\varphi_n^k\}$. The idea is that for each k, we will choose an appropriate $n(k)$ and set $\hat\varphi^k := \varphi_{n(k)}^k$. We need to choose a suitable $n(k)$ such that $n(k) \to \infty$ and (3.13) holds for $\varphi_{n(k)}^k$ when k is large enough. To this end we introduce an "inverse" of k as

$$
n(k) := \min\{j \in \mathbb{N} : k < k_0(j + 1)\}.
$$

$n(k)$ is well-defined, non-decreasing and tends to ∞ as $k \to \infty$. From the definition of $n(k)$ we see that if $k \geq k_0(2)$ then $n(k) \geq 2$ and $k_0(n(k)) \leq k < k_0(n(k) + 1)$ (otherwise $n(k)$ is not the minimum). Thus by (3.13) and definition of $\hat\varphi^k$ we have

for all $k \geq k_0(2)$,

$$
\begin{cases}
\|\hat{\varphi}^k - \overline{\varphi}\|_{L^2(\Omega)} \leq \min\left\{\dfrac{1}{n(k)}, \dfrac{1}{Mn(k)}\right\}, \\[2mm]
\begin{aligned}
J^{\lambda_k}(\hat{\varphi}^k, \Omega) &= J^{\lambda_k}(\varphi^k_{n(k)}, \Omega) \\
&\leq \left(1 + \dfrac{1}{n(k)}\right)\left(J^0(\overline{\varphi}, \Omega) + \dfrac{\beta+1}{n(k)}\right) \\
&\quad + \dfrac{1}{n(k)}\left(2\|\overline{\varphi}\|_{L^2(\Omega)} + \dfrac{1}{n(k)} + 1\right) + \dfrac{2}{n(k)}.
\end{aligned}
\end{cases}
$$

Sending $k \to \infty$ we get

$$
\begin{cases}
\lim\limits_{k\to\infty} \|\hat{\varphi}^k - \bar{\varphi}\|_{L^2(\Omega)} = 0, \\[2mm]
\limsup\limits_{k\to\infty} J^{\lambda_k}(\hat{\varphi}^k, \Omega) \leq J^0(\bar{\varphi}, \Omega).
\end{cases}
$$

On the other hand, by Theorem 3.4,

$$
J^0(\bar{\varphi}, \Omega) \leq \liminf_{k\to\infty} J^{\lambda_k}(\hat{\varphi}^k, \Omega)
$$

and thus we can conclude that $\hat{\varphi}^k \to \bar{\varphi}$ strongly in $L^2(\Omega)$ and $J^{\lambda_k}(\hat{\varphi}^k, \Omega) \to J^0(\bar{\varphi}, \Omega)$. Moreover, by the definitions of $\varphi^k_\varepsilon, \hat{\varphi}^k$, we also have $\hat{\varphi}^k \in H^1(\Omega)$ and $\hat{\varphi}^k - w^k$ has compact support in Ω.

Now set $\tilde{\varphi}^k := |\hat{\varphi}^k|$. Then $\tilde{\varphi}^k \in H^1(\Omega), \tilde{\varphi}^k \geq 0, \tilde{\varphi}^k = w^k$ in $\Omega^{\mathsf{c}} \supset K^{\lambda_k}$ for k large enough, and thus $\tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ for k large enough. Moreover, following the argument in the proof of [30, Lemma 4.5], $\tilde{\varphi}^k \to \bar{\varphi}$ in $L^2(\Omega)$ and $J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \to J^0(\bar{\varphi}, \Omega)$.

Since $w^k, \tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ and $\mathrm{supp}(\tilde{\varphi}^k - w^k) \subset \Omega$, by (2.13) and integration by parts formula we have

$$
a^{\lambda_k}_\Omega(w^k, \tilde{\varphi}^k - w^k) \geq -\lambda_k^{\frac{2-n}{n}}\left\langle u^{\lambda_k}_t, \tilde{\varphi}^k - w^k\right\rangle_\Omega + \left\langle -\dfrac{1}{g^{\lambda_k}}, \tilde{\varphi}^k - w^k\right\rangle_\Omega.
$$

The inequality $a^{\lambda_k}(u, v-u) \leq \frac{1}{2}J^{\lambda_k}(v) - \frac{1}{2}J^{\lambda_k}(u)$ for any u, v implies

$$
\frac{1}{2}J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \geq \frac{1}{2}J^{\lambda_k}(w^k, \Omega) - \lambda_k^{\frac{2-n}{n}}\left\langle u^{\lambda_k}_t, \phi^k\right\rangle_\Omega + \left\langle -\dfrac{1}{g^{\lambda_k}}, \phi^k\right\rangle_\Omega,
$$

where $\phi^k := \tilde{\varphi}^k - w^k \to \phi$ in $L^2(\Omega)$. Taking $\liminf$ as $k \to \infty$ and using the fact that $u^{\lambda_k}_t$ is bounded give

(3.14)
$$
\frac{1}{2}J^0(\overline{\varphi}, \Omega) \geq \frac{1}{2}J^0(\overline{w}, \Omega) + \langle -L, \phi\rangle_\Omega.
$$

This holds for any $\phi \in W_1$ and therefore also for $\delta\phi$, where $0 < \delta < 1$. Replacing ϕ in (3.14) by $\delta\phi$ we have

$$
\frac{1}{2}J^0(\bar{w} + \delta\phi, \Omega) \geq \frac{1}{2}J^0(\bar{w}, \Omega) + \langle -L, \delta\phi\rangle
$$

$$
\Leftrightarrow \frac{1}{2}\left[J^0(\bar{w}, \Omega) + 2\delta q_\Omega(\bar{w}, \phi) + \delta^2 J^0(\phi)\right] \geq \frac{1}{2}J^0(\bar{w}, \Omega) + \langle -L, \delta\phi\rangle.
$$

Dividing both sides by δ and sending $\delta \to 0$ we obtain

$$
q_\Omega(\overline{w}, \phi) \geq \langle -L, \phi\rangle_\Omega.
$$

Since $\mathrm{supp}\,\phi \in \Omega$, we conclude that (3.7) holds in $\mathbb{R}^n$.

Now take $\psi \in W_2$. As above, we assume that ψ has a compact support contained in Ω, and without loss of generality we can also assume that $0 \le \psi \le 1, \psi = 0$ on $B_\varepsilon(0)$ (otherwise consider $\frac{\psi}{\max_{\mathbb{R}^n} \psi}$ instead). Since $\psi \in W_2$ then $\psi \overline{w} \in W_1$ and (3.7) holds for $\psi \overline{w}$, we have $q(\overline{w}, \psi \overline{w}) \ge \langle -L, \psi \overline{w} \rangle$. For the reverse inequality, define $\overline{\varphi} := (1 - \psi) \overline{w} \in H^1(\Omega)$. Arguing as before, we can choose $\tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ such that $\tilde{\varphi}^k \to \overline{\varphi}$ in $L^2(\Omega), J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \to J^0(\overline{\varphi}, \Omega)$. Again, since $w^k, \tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$, by (2.13) and the inequality $a^{\lambda_k}(u, v - u) \le \frac{1}{2} J^{\lambda_k}(v) - \frac{1}{2} J^{\lambda_k}(u)$ for any u, v we have

$$\frac{1}{2} J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \ge \frac{1}{2} J^{\lambda_k}(w^k, \Omega) - \lambda_k^{\frac{2-n}{n}} \left\langle u_t^{\lambda_k}, \tilde{\varphi}^k - w^k \right\rangle_\Omega + \left\langle -\frac{1}{g^{\lambda_k}}, \tilde{\varphi}^k - w^k \right\rangle_\Omega .$$

Taking $\liminf$ as $k \to \infty$ and arguing the same as in the proof of (3.7) we get

$$q_\Omega(\overline{w}, \overline{\varphi} - \overline{w}) \ge \langle -L, \overline{\varphi} - \overline{w} \rangle_\Omega$$
$$\Leftrightarrow \quad -q_\Omega(\overline{w}, \psi \overline{w}) \ge -\langle -L, \psi \overline{w} \rangle_\Omega$$
$$\Leftrightarrow \quad q_\Omega(\overline{w}, \psi \overline{w}) \le \langle -L, \psi \overline{w} \rangle_\Omega .$$

Thus $q(\overline{w}, \psi \overline{w}) = \langle -L, \psi \overline{w} \rangle$ for every $\psi \in W_2$. $\qquad\qquad \square$

This completes the proof of Theorem 3.1. $\qquad\qquad \square$

4. UNIFORM CONVERGENCE OF RESCALED VISCOSITY SOLUTIONS AND FREE BOUNDARIES

In this final section, we establish the convergence of the rescaled viscosity solutions v^λ of the Stefan problem (1.1) and their free boundaries. The proof is based on viscosity arguments showing that the half-relaxed limits of v^λ in $\{|x| \ne 0, t \ge 0\}$ defined as

$$v^*(x, t) = \limsup_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y, s), \qquad v_*(x, t) = \liminf_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y, s)$$

coincide and are the viscosity solution of the limit problem with a point source. We have the following result, which is similar to [38, Theorem 4.2].

Theorem 4.1. *Let $n \ge 3$ and $V = V_{C_*, L}$ be the solution of Hele-Shaw problem with a point source (2.27) with the constant C_* from Lemma 2.23 and $L = \left\langle \frac{1}{g} \right\rangle$ as in Lemma 3.2. The rescaled viscosity solution v^λ of the Stefan problem (1.1) converges locally uniformly to $V = V_{C_*, \left\langle \frac{1}{g} \right\rangle}$ in $(\mathbb{R}^n \setminus \{0\}) \times [0, \infty)$ as $\lambda \to \infty$ and*

$$v_* = v^* = V.$$

Moreover, the rescaled free boundary $\{\Gamma(v^\lambda)\}_\lambda$ converges to $\Gamma(V)$ locally uniformly with respect to the Hausdorff distance.

All the viscosity arguments used in [38, Section 4] can be applied in our anisotropic case with some minor adaptations. Therefore, we will omit some of the proofs and refer to [29, 30, 36, 38] for more details. Let us give a brief review of the ideas in the spirit of [38, Section 4] as follows.

(1) We first prove the convergence of the rescaled viscosity solution and its free boundary under the condition (1.6).
   - By the regularity of the initial data v_0 as in (1.6), we deduce a weak monotonicity of the solution v.

- Using the weak monotonicity and pointwise comparison principle arguments, we then show the convergence for regular initial data.

(2) For general initial data, we will find regular upper and lower approximations of the initial data satisfying (1.6) and use the comparison principle together with the uniqueness of the limit solution to reach the conclusion.

We will state the necessary results here with remarks on the adaptations for the anisotropic case.

4.1. **Some necessary technical results.** First, we have the correct singularity of v^* and v_* at the origin, which can be established similarly to the proof of Lemma 3.7.

Lemma 4.2 (cf. [38, Lemma 4.3], v^* and v_* behave as V at the origin). *The functions v^*, v_* have a singularity at 0 with*

$$(4.1) \qquad \lim_{|x|\to 0+} \frac{v_*(x,t)}{V(x,t)} = 1, \qquad \lim_{|x|\to 0+} \frac{v^*(x,t)}{V(x,t)} = 1, \quad for\ t > 0.$$

Proof. Argue as in the proof of Lemma 3.7. $\qquad\qquad\qquad\qquad\qquad\square$

We will also make use of an uniform estimate on u^λ and the convergence of boundary points deduced from the convergence of variational solutions.

Lemma 4.3 (cf. [29, Lemma 3.1]). *There exists constant $C > 0$ independent of λ such that for every $x_0 \in \overline{\Omega_{t_0}(u^\lambda)}$ and $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ for some r, we have*

$$\sup_{x\in\overline{B_r(x_0)}} u^\lambda(x,t_0) > Cr^2$$

for every λ.

Proof. We will prove the statement for $x_0 \in \Omega_{t_0}(u^\lambda)$ first, the results then follows by continuity of u^λ. Since $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ then u^λ satisfies

$$\lambda^{\frac{2-n}{n}} u_t^\lambda - \mathcal{L}^\lambda u^\lambda = -\frac{1}{g^\lambda} \text{ in } \{u^\lambda > 0\} \cap (B_r(x_0) \times \{t = t_0\}).$$

Since $u_t^\lambda \geq 0$ and $-\frac{1}{g} \leq -\frac{1}{M}$ then $-\mathcal{L}^\lambda u^\lambda \leq -\frac{1}{M} =: -C_0$ in $\{u^\lambda > 0\} \cap (B_r(x_0) \times \{t = t_0\})$.

Define

$$w^\lambda(x) = u^\lambda(x,t_0) - \frac{C_0}{n} h^\lambda(x - x_0)$$

where $h^\lambda(x)$ is the barrier with quadratic growth corresponding to elliptic operator $\mathcal{L}^\lambda$ introduced in Section 2.4.2. We have $\{w^\lambda > 0\} \cap B_r(x_0) \subset \{u^\lambda > 0\} \cap \{t = t_0\}$ and therefore, for all λ,

$$-\mathcal{L}^\lambda w^\lambda \leq 0 \text{ in } \{w^\lambda > 0\} \cap B_r(x_0).$$

We see that $w^\lambda(x_0) > 0$. Hence the maximum of w^λ in $\overline{B_r(x_0)}$ is positive and by the maximum principle, w^λ attains the maximum on the boundary $\{w^\lambda > 0\} \cap \partial B_r(x_0)$ and therefore

$$\sup_{\overline{B_r(x_0)}} u^\lambda(x,t_0) \geq \sup_{|x-x_0|=r} u^\lambda(x,t_0) > \inf_{|x-x_0|=r} \frac{C_0}{n} h^\lambda(x - x_0).$$

By the quadratic growth of h^λ, where the coefficients on the growth rate only depend on the elliptic constants, we have

$$\sup_{\overline{B_r(x_0)}} u^\lambda(x, t_0) \geq Cr^2,$$

for some constant C which does not depend on λ. $\qquad\square$

Lemma 4.4 (cf. [30, Lemma 5.4]). *Suppose that $(x_k, t_k) \in \{u^{\lambda_k} = 0\}$ and $(x_k, t_k, \lambda_k) \to$* ▮ *$(x_0, t_0, \infty)$. Let $U = U_{C_*, L}$ be the limit function as in Theorem 3.1. Then:*

a) $U(x_0, t_0) = 0$,

b) If $x_k \in \Gamma_{t_k}(u^{\lambda_k})$ then $x_0 \in \Gamma_{t_0}(U)$,

Proof. See the proof of [30, Lemma 5.4]. $\qquad\square$

A weak monotonicity in time of the solution of the Stefan problem (1.1) is given by the following lemma.

Lemma 4.5 (cf. [38, Lemma 4.7, Lemma 4.8], Weak monotonicity). *Let u be the solution of the variational problem (2.1), and v be the associated viscosity solution of the Stefan problem. Suppose that v_0 satisfies (1.6). Then there exist $C \geq 1$ independent of x and t such that*

(4.2) $\qquad v_0(x) \leq Cv(x, t)$ *and* $u(x, t) \leq Ctv(x, t)$ *in* $\mathbb{R}^n \setminus K \times [0, \infty)$.

Proof. Following the same arguments as in [38, Lemma 4.7, Lemma 4.8], we obtain (4.2) simply by using elliptic operator $\mathcal{L}$ instead of the Laplace operator. $\qquad\square$

Lemma 4.3 and Lemma 4.5 automatically give us a crucial uniform lower estimate on v^λ and allow us to show the relationship between v_*, v^* and V.

Corollary 4.6. *There exists a constant $C_1 = C_1(n, M)$ such that if $(x_0, t_0) \in \Omega(v^\lambda)$ and $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$, we have*

$$\sup_{B_r(x_0)} v^\lambda(x, t_0) \geq \frac{C_1 r^2}{t_0}.$$

Lemma 4.7. *Let v be the viscosity solution of (1.1) and v^λ be its rescaling. Then the following statements hold.*

i) $v^*(\cdot, t)$ is a subsolution of (2.10) in $\mathbb{R}^n \setminus \{0\}$ and $v_*(\cdot, t)$ is a supersolution of (2.10) in $\Omega_t(v_*) \setminus \{0\}$ in viscosity sense.

ii) $\Omega(V) \subset \Omega(v_*)$ and in particular $v_* \geq V$.

iii) $\Gamma(v^*) \subset \Gamma(V)$.

Proof. i) follows from standard viscosity arguments with noting that we can take a sequence of test functions for rescaled elliptic equation that converges to the test function for (2.10) by classical homogenization results.

ii) See [30, Lemma 5.5], the conclusion holds by i), Lemma 4.2 and Lemma 4.5.

iii) See [30, Lemma 5.6 ii]. $\qquad\square$

Proof of Theorem 4.1.

Proof. We follow the proof of [38, Theorem 4.2]; see [38] for more details.

Step 1. We first show the convergence results for the problem with the initial data satisfying (1.6) using the weak monotonicity in time of the solution, Lemma 4.5, and its consequences.

By Lemma 4.7, the correct singularity of v^* from Lemma 4.2 and the comparison principle for elliptic equation (2.10) we have

$$V(x,t) \leq v_*(x,t) \leq v^*(x,t) \leq V_{C_*+\varepsilon,\langle\frac{1}{g}\rangle}(x,t).$$

Let $\varepsilon \to 0$ we obtain $v_* = v^* = V$ by continuity and in particular, $\Gamma(v_*) = \Gamma(v^*) = \Gamma(V)$.

Now we show the locally uniform convergence of the free boundaries with respect to the Hausdorff distance. To simplify the notation, we fix $0 < t_1 < t_2$ and define

$$\Gamma^\lambda := \Gamma(v^\lambda) \cap \{t_1 \leq t \leq t_2\}, \qquad \Gamma^\infty := \Gamma(V) \cap \{t_1 \leq t \leq t_2\}.$$

The result will follow if we show that for all $\delta > 0$, there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$,

(4.3)
$$\text{dist}((x_0,t_0),\Gamma^\infty) < \delta \text{ for all } (x_0,t_0) \in \Gamma^\lambda, \text{ and}$$
$$\text{dist}((x_0,t_0),\Gamma^\lambda) < \delta \text{ for all } (x_0,t_0) \in \Gamma^\infty.$$

A contradiction argument as in the proofs of [36, Theorem 7.1] and [38, Theorem 4.2], using Lemma 4.4 above yields the existence of λ_0 for the first inequality. Hence the main task now is to show the existence of λ_0 for the second inequality in (4.3). Note that we only need to show this pointwise. The result then follows from the compactness of Γ^∞. Suppose that there exists $\delta > 0, (x_0,t_0) \in \Gamma^\infty$ and $\{\lambda_k\}, \lambda_k \to \infty$, such that $\text{dist}((x_0,t_0),\Gamma^{\lambda_k}) \geq \dfrac{\delta}{2}$ for all k. Then there exists $r > 0$ such that after passing to a subsequence if necessary, we can assume that $D_r(x_0,t_0) := B(x_0,r) \times [t_0 - r, t_0 + r]$ satisfies either

(4.4)
$$D_r(x_0,t_0) \subset \{v^{\lambda_k} = 0\}, \quad \text{for all } k$$

or,

(4.5)
$$D_r(x_0,t_0) \subset \{v^{\lambda_k} > 0\}, \quad \text{for all } k.$$

But (4.4) clearly implies $V = v_* = 0$ in $D_r(x_0,t_0)$, contradicting $(x_0,t_0) \in \Gamma^\infty$. Thus we assume (4.5). Following [38], to handle Harnack's inequality for a parabolic equation that becomes elliptic in the limit, we rescale time as

$$w^k(x,t) := v^{\lambda_k}(x, \lambda_k^{\frac{2-n}{n}} t).$$

Then $w^k > 0$ in $D_r^w(x_0,t_0) := B(x_0,r) \times [\lambda_k^{\frac{n-2}{n}}(t_0-r), \lambda_k^{\frac{n-2}{n}}(t_0+r)]$ and w^k satisfies $w_t^k - \mathcal{L}^\lambda w^k = 0$ in $D_r^w(x_0,t_0)$. Since $\lambda_k^{\frac{n-2}{n}} \to \infty$ as $k \to \infty$ then for any fixed $\tau > 0$, there exists λ_0 such that $\tau < \lambda_k^{\frac{n-2}{n}} \frac{r}{4}$ for all $\lambda_k \geq \lambda_0$. Now applying Harnack's inequality for the parabolic equation $w_t^k - \mathcal{L}^\lambda w^k = 0$ we have for a fixed $\tau > 0$, there exists a constant $C_1 > 0$ such that for each $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$ and all λ_k such that $\tau < \lambda_k^{\frac{n-2}{n}} \frac{r}{4}$ we have

$$\sup_{B(x_0,\frac{r}{2})} w^k(\cdot, \lambda_k^{\frac{n-2}{n}} t - \tau) \leq C_1 \inf_{B(x_0,\frac{r}{2})} w^k(\cdot, \lambda_k^{\frac{n-2}{n}} t).$$

As noted in [38, Theorem 4.2], for the isotropic case, the constant C_1 of Harnack's inequality can be taken not depending on λ_k. For the anisotropic case, this constant also depends on the elliptic constants of operator $\mathcal{L}^{\lambda_k}$. However the rescaling of the

operator does not change the elliptic constants. Thus C_1 can be taken independent of λ_k. By this inequality and Corollary 4.6 we have

$$\frac{C_2 r^2}{t - \lambda_k^{\frac{2-n}{n}} \tau} \leq \sup_{B\left(x_0, \frac{r}{2}\right)} v^{\lambda_k}\left(\cdot, t - \lambda_k^{\frac{2-n}{n}} \tau\right) \leq C_1 \inf_{B\left(x_0, \frac{r}{2}\right)} v^{\lambda_k}(\cdot, t)$$

for all $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, $\lambda_k \geq \lambda_0$ large enough, where C_2 only depends on n, M. In the limit $\lambda_k \to \infty$, the uniform convergence of $\{v^{\lambda_k}\}$ to V implies $V > 0$ in $B(x_0, \frac{r}{2}) \times [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, which contradicts the assumption $(x_0, t_0) \in \Gamma^\infty \subset \Gamma(V)$. This concludes the proof of Theorem 4.1 when (4.2) holds.

Step 2. For general initial data, arguing as in step 2 of the proof of [38, Theorem 4.2], we are able to find upper and lower bounds for the initial data for which (4.2) holds. The comparison principle for viscosity solution of the Stefan problem (1.1) then yields the convergence since the limit function V is unique and does not depend on the initial data.

$\square$

Acknowledgments

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