LONG-TIME BEHAVIOR OF THE ONE-PHASE STEFAN PROBLEM IN PERIODIC AND RANDOM MEDIA

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ABSTRACT. We study the long-time behavior of solutions of the one-phase Stefan problem in inhomogeneous media in dimensions $n \geq 2$. Using the technique of rescaling which is consistent with the evolution of the free boundary, we are able to show the homogenization of the free boundary velocity as well as the locally uniform convergence of the rescaled solution to a self-similar solution of the homogeneous Hele-Shaw problem with a point source. Moreover, by viscosity solution methods, we also deduce that the rescaled free boundary uniformly approaches a sphere with respect to Hausdorff distance.

1. INTRODUCTION

We consider the one-phase Stefan problem in periodic and random media in a dimension $n \geq 2$. The aim of this paper is to understand the behavior of the solutions and their free boundaries when time $t \to \infty$.

Let $K \subset \mathbb{R}^n$ be a compact set with sufficiently regular boundary, for instance $\partial K \in C^{1,1}$, and assume that $0 \in \operatorname{int} K$. The one-phase Stefan problem (on an exterior domain) with inhomogeneous latent heat of phase transition is to find a function $v(x,t) : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ that satisfies the free boundary problem

(1.1)
$$\begin{cases} v_t - \Delta v = 0 & \text{in } \{v > 0\} \backslash K, \\ \quad\;\; v = 1 & \text{on } K, \\ \quad\;\; V_\nu = g(x)|Dv| & \text{on } \partial\{v > 0\}, \\ v(x, 0) = v_0 & \text{on } \mathbb{R}^n, \end{cases}$$

where D and Δ are respectively the spatial gradient and Laplacian, v_t is the partial derivative of v with respect to time variable t, V_ν is the normal velocity of the *free boundary* $\partial\{v > 0\}$. v_0 and g are given functions, see below. Note that the results in this paper can be trivially extended to general time-independent positive continuous boundary data, 1 is taken only to simplify the exposition.

The one-phase Stefan problem is a mathematical model of phase transitions between a solid and a liquid. A typical example is the melting of a body of ice maintained at temperature 0, in contact with a region of water. The unknowns are the temperature distribution v and its free boundary $\partial\{v(\cdot, t) > 0\}$, which models the ice-water interface. Given an initial temperature distribution of the water, the diffusion of heat in a medium by conduction and the exchange of latent heat will govern the system. In this paper, we consider an inhomogeneous medium where

the latent heat of phase transition, $L(x) = 1/g(x)$, and hence the velocity law depend on position. The related Hele-Shaw problem is usually referred to in the literature as the quasi-stationary limit of the one-phase Stefan problem when the heat operator is replaced by the Laplace operator. This problem typically describes the flow of an injected viscous fluid between two parallel plates which form the so-called Hele-Shaw cell, or the flow in porous media.

In this paper, we assume that the function g satisfies the following two conditions, which guarantee respectively the well-posedness of (1.1) and averaging behavior as $t \to \infty$:

(1) g is a Lipschitz function in $\mathbb{R}^n$, $m \leq g \leq M$ for some positive constants m and M.
(2) $g(x)$ has some averaging properties so that Lemma 3.1 applies, for instance, one of the following holds:
  (a) g is a $\mathbb{Z}^n$-periodic function,
  (b) $g(x,\omega) : \mathbb{R}^n \times A \to [m, M]$ is a stationary ergodic random variable over a probability space $(A, \mathcal{F}, P)$.

For a detailed definition and overview of stationary ergodic media, we refer to [17, 16] and the references therein.

Throughout most of the paper we will assume that the initial data v_0 satisfies

(1.2)
$$v_0 \in C^2(\overline{\Omega_0 \setminus K}), v_0 > 0 \text{ in } \Omega_0, v_0 = 0, \text{ on } \Omega_0^c := \mathbb{R}^n \setminus \Omega_0, \text{ and } v_0 = 1 \text{ on } K,$$
$$|Dv_0| \neq 0 \text{ on } \partial\Omega_0, \text{ for some bounded domain } \Omega_0 \supset K.$$

This will guarantee the existence of both the weak and viscosity solutions below and their coincidence, as well as the weak monotonicity (4.2). However, the asymptotic limit, Theorem 1.1, is independent of the initial data, and therefore the result applies to arbitrary initial data as long as the (weak) solution exists, satisfies the comparison principle, and the initial data can be approximated from below and from above by data satisfying (1.2). For instance, $v_0 \in C(\mathbb{R}^n)$, $v_0 = 1$ on K, $v_0 \geq 0$, supp v_0 compact is sufficient.

The Stefan problem (1.1) does not necessarily have a global classical solution in $n \geq 2$ as singularities of the free boundary might develop in finite time. The classical approach to define a generalized solution is to integrate v in time and introduce $u(x,t) := \int_0^t v(x, s)ds$ [1, 6, 9, 7, 21, 23, 22]. If v is sufficiently regular, then u solves the variation inequality

(1.3)
$$\begin{cases} u(\cdot, t) \in \mathcal{K}(t), \\ (u_t - \Delta u)(\varphi - u) \geq f(\varphi - u) \text{ a.e } (x, t) \text{ for any } \varphi \in \mathcal{K}(t), \end{cases}$$

where $\mathcal{K}(t)$ is a suitable functional space specified later in Section 2.2 and f is

(1.4)
$$f(x) = \begin{cases} v_0(x), & v_0(x) > 0, \\ -\dfrac{1}{g(x)}, & v_0(x) = 0. \end{cases}$$

This parabolic inequality always has a global unique solution $u(x, t)$ for initial data satisfying (1.2) [9, 21, 23, 22]. The corresponding time derivative $v = u_t$, if it exists, is then called a *weak solution* of the Stefan problem (1.1). The main advantage of this definition is that the powerful theory of variational inequalities can be applied for the study of the Stefan problem, and as was observed in [20, 15, 16] yields homogenization of (1.3).

More recently, the notion of viscosity solutions of the Stefan problem was introduced and well-posedness was established by Kim [11]. Since this notion relies on the comparison principle instead of the variational structure, it allows for more general, fully nonlinear parabolic operators and boundary velocity laws. Moreover, the pointwise viscosity methods seem more appropriate for studying the behavior of the free boundaries. The natural question whether the weak and viscosity solutions coincide was answered positively by Kim and Mellet [16] whenever the weak solution exists. In this paper we will use the strengths of both the weak and viscosity solutions to study the behavior of the solution and its free boundary for large times.

The homogeneous version of this problem, i.e, when $g \equiv const$, was studied by Quirós and Vázques in [19]. They obtained the result on the long-time convergence of weak solution of the one-phase Stefan problem to the self-similar solution of the Hele-Shaw problem. The homogenization of this type of problem was considered by Rodrigues in [20] and by Kim-Mellet in [15, 16]. The long-time behavior of solution of the Hele-Shaw problem was studied in detail by the first author in [17]. In particular, the rescaled solution of the inhomogeneous Hele-Shaw problem converges to the self-similar solution of the Hele-Shaw problem with a point-source, formally

$$(1.5) \qquad \begin{cases} -\Delta v = C\delta & \text{in } \{v > 0\}, \\ v_t = \dfrac{1}{\langle 1/g \rangle}|Dv|^2 & \text{on } \partial\{v > 0\}, \\ v(\cdot, 0) = 0, \end{cases}$$

where δ is the Dirac δ-function, C is a constant depending on K and n, and the constant $\langle 1/g \rangle$ will be properly defined later. Moreover, the rescaled free boundary uniformly approaches a sphere.

Here we extend the convergence result to the Stefan problem in the inhomogeneous medium. Since the asymptotic behavior of radially symmetric solutions of the Hele-Shaw and the Stefan problem are similar and the solutions are bounded, we can take the limit $t \to \infty$ and obtain the convergence for rescaled solutions and their free boundaries. However, solutions of the Hele-Shaw problem have a very useful monotonicity in time, which is missing in the Stefan problem. This makes some steps more difficult. We instead take advantage of a weak monotonicity property (4.2), which holds for regular initial data satisfying (1.2) and then show the convergence result for general initial data using the uniqueness of the limit and the comparison principle. Moreover, the heat operator is not invariant under the rescaling, unlike the Laplace operator. The rescaled parabolic equation becomes elliptic when $\lambda \to \infty$, which causes some issues when applying parabolic Harnack's inequality, for instance. Following [19, 17] we use the natural rescaling of solutions of the form

$$v^\lambda(x, t) := \lambda^{(n-2)/n} v(\lambda^{1/n} x, \lambda t) \qquad\qquad \text{if } n \geq 3,$$

and the corresponding rescaling for variational solutions

$$u^\lambda(x, t) := \lambda^{-2/n} u(\lambda^{1/n}, \lambda t) \qquad\qquad \text{if } n \geq 3$$

(see Section 2.4 for $n = 2$). Then the rescaled viscosity solution satisfies the free boundary velocity law

$$V_\nu^\lambda = g(\lambda^{1/n} x)|Dv^\lambda|.$$

Heuristically, if g has some averaging properties, such as in condition (2), the free boundary velocity law should homogenize as $\lambda \to \infty$. Since the latent heat of phase transition $1/g$ should average out, the homogenized velocity law will be

$$V_\nu = \frac{1}{\langle 1/g \rangle} |Dv|,$$

where $\langle 1/g \rangle$ represents the "average" of $1/g$. More precisely, the quantity $\langle 1/g \rangle$ is the constant in the subadditive ergodic theorem such that

$$\int_{\mathbb{R}^n} \frac{1}{g(\lambda^{1/n} x, \omega)} u(x) dx \to \int_{\mathbb{R}^n} \left\langle \frac{1}{g} \right\rangle u(x) dx \text{ for all } u \in L^2(\mathbb{R}^n), \text{ for a.e. } \omega \in A.$$

In the periodic case, it is just the average of $1/g$ over one period. Since we always work with $\omega \in A$ for which the convergence above holds, we omit it from the notation in the rest of the paper.

This yields the first main result of this paper, Theorem 3.2, on the homogenization of the obstacle problem (1.3) for the rescaled solutions, with the correct singularity of the limit function at the origin, and therefore the locally uniform convergence of variational solutions. To prove the second main result in Theorem 4.2 on the locally uniform convergence of viscosity solutions and their free boundaries, we use pointwise viscosity solution arguments. In summary, we will show the following theorem.

Theorem 1.1. *For almost every $\omega \in A$, the rescaled viscosity solution v^λ of the Stefan problem (1.1) converges locally uniformly to the unique self-similar solution V of the Hele-Shaw problem (1.5) in $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$ as $\lambda \to \infty$, where C depends only on n, the set K and the boundary data 1. Moreover, the rescaled free boundary $\partial\{(x,t) : v^\lambda(x,t) > 0\}$ converges to $\partial\{(x,t) : V(x,t) > 0\}$ locally uniformly with respect to the Hausdorff distance.*

It is a natural question to consider more general linear divergence form operators $\sum_{i,j} \partial_{x_i}(a_{ij}(x)\partial_{x_j} \cdot)$ instead of the Laplacian in (1.1) so that the variational structure is preserved. This was indeed the setting considered in [16], with $g \equiv 1$ and appropriate free boundary velocity law adjusted for the operator above. In the limit $\lambda \to \infty$, we expect that the rescaled solutions v^λ to converge to the unique solution of the Hele-Shaw type problem with a point source with the homogenized non-isotropic operator with coefficients $\bar{a}_{i,j}$. This question is a topic of ongoing work.

Context and open problems. In recent years, there have been significant developments in the homogenization theory of partial differential equations like Hamilton-Jacobi and second order fully nonlinear elliptic and parabolic equations that have been made possible by the improvements of the viscosity solutions techniques, see for instance the classical [8, 24, 5, 4] to name a few.

A common theme of these results is finding (approximate) correctors and use the perturbed test function method to establish the homogenization result in the periodic case, or use deeper properties in the random case, such as the variational structure of the Hamilton-Jacobi equations or the strong regularity results for elliptic and parabolic equations, including the ABP inequality.

One of the goals of this paper is to illustrate the powerful combination of variational and viscosity solution techniques for some free boundary problems that

have a variational structure. By viscosity solution techniques we mean specifically pointwise arguments using the comparison principle.

Unfortunately, when the variational structure is lost, for instance, when the free boundary velocity law is more general as in the problem with contact angle dynamics $V_\nu = |Dv| - g(x)$ so that the motion is non-monotone [13, 14], or even simple time-dependence $V_\nu = g(x,t)|Dv|$ [18], the comparison principle is all that is left. Even in the periodic case, the classical correctors as solutions of a cell problem are not available. This is in part the consequence of the presence of the free boundary on which the operator is strongly discontinuous. [12, 13, 18] use a variant of the idea that appeared in [5] to replace the correctors by solutions of certain obstacle problems. However, the analysis of these solutions requires rather technical pointwise arguments since there are almost no equivalents of the regularity estimates for elliptic equations. An important tool in [18] to overcome this was the large scale Lipschitz regularity of the free boundaries of the obstacle problem solutions (called cone flatness there) that allows for the control of the oscillations of the free boundary in the homogenization limit.

For the reasons above, the homogenization of free boundary problems is rather challenging and there are still many open problems. Probably the most important one is the homogenization of free boundary problems of the Stefan and Hele-Shaw type that do not admit a variational structure, such as those mentioned above, in random environments. Currently there is no known appropriate stationary subadditive quantity to which we could apply the subadditive ergodic theorem to recover the homogenized free boundary velocity law, for instance. Other tools like concentration inequalities have so far not yielded an alternative.

Another important problem are the optimal convergence rates of the free boundaries in the Hausdorff distance. The techniques used in this paper do not provide this information, however viscosity techniques were used to obtain non-optimal algebraic convergence rates in [14]. It is an interesting question what the optimal rate in the periodic case is, even for problems like (1.1). The large scale Lipschitz estimate from [18] could possibly directly give only $\varepsilon|\log \varepsilon|^{1/2}$-rate for velocity law with $g(x/\varepsilon)$, but there are some indications that a rate ε might be possible.

Outline. The paper is organized as follows: In Section 2, we recall the definitions and well-known results for weak and viscosity solutions. We also introduce the rescaling and state some results for radially symmetric solutions. In Section 3, we recall the limit obstacle problem and prove the locally uniform convergence of rescaled variational solutions. In Section 4, we focus on treating the locally uniform convergence of viscosity solutions and their free boundaries.

2. Preliminaries

2.1. **Notation.** For a set A, A^c is its complement. Given a nonnegative function v, we will use notations for its positive set and free boundary of v,

$$\Omega(v) := \{(x,t) : v(x,t) > 0\}, \qquad\qquad \Gamma(v) := \partial\Omega(v),$$

and for fixed time t, $\qquad \Omega_t(v) := \{x : v(x,t) > 0\}, \qquad \Gamma_t(v) := \partial\Omega_t(v).$

$(f)_+$ is the positive part of f: $(f)_+ = \max(f, 0)$.

2.2. **Weak solutions.** Let $v(x,t)$ be a classical solution of the Stefan problem (1.1). Fix $R, T > 0$ and set $B = B_R(0)$, $D = B \backslash K$. Following [9] it can be shown that, if R is large enough (depending on T), then the function $u(x,t) := \int_0^t v(x,s)ds$ solves the following variational problem: Find $u \in L^2(0,T;H^2(D))$ such that $u_t \in L^2(0,T;L^2(D))$ and

$$(2.1) \quad \begin{cases} \qquad u(\cdot,t) \in \mathcal{K}(t), \qquad 0 < t < T, \\ (u_t - \Delta u)(\varphi - u) \geq f(\varphi - u), \quad \text{a.e } (x,t) \in B \times (0,T) \text{ for any } \varphi \in \mathcal{K}(t), \\ \qquad u(x,0) = 0 \text{ in } D. \end{cases}$$

Here we set $\mathcal{K}(t) = \{\varphi \in H^1(D), \varphi \geq 0, \varphi = 0 \text{ on } \partial B, \varphi = t \text{ on } K\}$ and f was defined in (1.4). We use the standard notation for Sobolev spaces H^k, $W^{k,p}$. Note that $u(x,t)$ is independent of the choice of B as long as R is large enough [16, Lemma 3.6]. If v is a classical solution of (1.1) then u is solution of (2.1), but the inverse statement is not valid in general. However, we have the following result [9, 21].

Theorem 2.1 (Existence and uniqueness of variational problem). *If v_0 satisfies (1.2), then the problem (2.1) has a unique solution satisfying*

$$u \in L^\infty(0,T;W^{2,p}(D)), \qquad 1 \leq p \leq \infty,$$
$$u_t \in L^\infty(D \times (0,T)),$$

and

$$\begin{cases} \qquad u_t - \Delta u \geq f & \text{for a.e. } (x,t) \in \{u \geq 0\}, \\ u(u_t - \Delta u - f) = 0 & \text{a.e in } D \times (0,\infty). \end{cases}$$

We will thus say that if u is a solution of (2.1), then u_t is a *weak solution* of the corresponding Stefan problem (1.1). The theory of variational inequalities for an obstacle problem is well developed, for more details, we refer to [9, 21, 15]. We now collect some useful results on the weak solutions from [9, 21].

Proposition 2.2. *The unique solution u of (2.1) satisfies*

$$0 \leq u_t \leq C \text{ a.e } D \times (0,T),$$

where C is a constant depending on f. In particular, u is Lipschitz with respect to t and u is $C^\alpha(D)$ with respect to x for all $\alpha \in (0,1)$. Furthermore, if $0 \leq t < s \leq T$, then $u(\cdot,t) < u(\cdot,s)$ in $\Omega_s(u)$ and also $\Omega_0 \subset \Omega_t(u) \subset \Omega_s(u)$.

Lemma 2.3 (Comparison principle for weak solutions). *Suppose that $f \leq \hat{f}$. Let $u, \hat{u}$ be solutions of (2.1) for respective $f, \hat{f}$. Then $u \leq \hat{u}$, moreover,*

$$\theta \equiv \frac{\partial u}{\partial t} \leq \frac{\partial \hat{u}}{\partial t} \equiv \hat{\theta}.$$

Remark 2.4. Regularity of θ and its free boundary has been studied quite extensively, including Caffarelli and Friedman (see [2, 3, 10]). It is known that a weak solution is classical as long as $\Gamma_t(u)$ has no singularity. The smoothness criterion (see [2, 10], [19, Proposition 2.4]) immediately leads to the following corollary.

Corollary 2.5. *Radial weak solutions of the Stefan problem (1.1) are smooth classical solutions.*

2.3. **Viscosity solutions.** The second notion of solutions we will use are the viscosity solutions introduced in [11]. First, for any nonnegative function $w(x,t)$ we define the semicontinuous envelopes

$$w_\star(x,t) := \liminf_{(y,s)\to(x,t)} w(y,s), \qquad w^\star(x,t) := \limsup_{(y,s)\to(x,t)} w(y,s).$$

We will consider solutions in the space-time cylinder $Q := (\mathbb{R}^n\backslash K) \times [0,\infty)$.

Definition 2.6. A nonnegative upper semicontinuous function $v(x,t)$ defined in Q is a viscosity subsolution of (1.1) if the following hold:

a) For all $T \in (0,\infty)$, the set $\overline{\Omega(v)} \cap \{t \leq T\} \cap Q$ is bounded.
b) For every $\phi \in C_{x,t}^{2,1}(Q)$ such that $v - \phi$ has a local maximum in $\overline{\Omega(v)} \cap \{t \leq t_0\} \cap Q$ at (x_0,t_0), the following holds:
   i) If $v(x_0,t_0) > 0$, then $(\phi_t - \Delta\phi)(x_0,t_0) \leq 0$.
   ii) If $(x_0,t_0) \in \Gamma(v), |D\phi(x_0,t_0)| \neq 0$ and $(\phi_t - \Delta\phi)(x_0,t_0) > 0$, then

$$(2.2) \qquad (\phi_t - g(x_0)|D\phi|^2)(x_0,t_0) \leq 0.$$

Analogously, a nonnegative lower semicontinuous function $v(x,t)$ defined in Q is a viscosity supersolution if (b) holds with maximum replaced by minimum, and with inequalities reversed in the tests for ϕ in (i–ii). We do not need to require (a).

Now let v_0 be a given initial condition with positive set Ω_0 and free boundary $\Gamma_0 = \partial\Omega_0$, we can define viscosity subsolution and supersolution of (1.1) with corresponding initial data and boundary data.

Definition 2.7. A viscosity subsolution of (1.1) in Q is a viscosity subsolution of (1.1) in Q with initial data v_0 and boundary data 1 if:

a) v is upper semicontinuous in $\bar{Q}, v = v_0$ at $t = 0$ and $v \leq 1$ on K,
b) $\overline{\Omega(v)} \cap \{t = 0\} = \overline{\{x : v_0(x) > 0\}} \times \{0\}$.

A viscosity supersolution is defined analogously by requiring (a) with v lower semicontinuous and $v \geq 1$ on Γ. We do not need to require (b).

And finally we can define viscosity solutions.

Definition 2.8. The function $v(x,t)$ is a viscosity solution of (1.1) in Q (with initial data v_0 and boundary data 1) if v is a viscosity supersolution and $v^\star$ is a viscosity subsolution of (1.1) in Q (with initial data v_0 and boundary data 1).

Remark 2.9. By standard argument, if v is the classical solution of (1.1) then it is a viscosity solution of that problem in Q with initial data v_0 and boundary data 1.

The existence and uniqueness of a viscosity solution as well as its properties have been studied in great detail in [11]. One important feature of viscosity solutions is that they satisfy a comparison principle for "strictly separated" initial data.

One of the main tools we will use in this paper is the following coincidence of weak and viscosity solutions from [16].

Theorem 2.10 (cf. [16, Theorem 3.1]). *Assume that v_0 satisfies (1.2). Let $u(x,t)$ be the unique solution of (2.1) in $B \times [0,T]$ and let $v(x,t)$ be the solution of*

$$(2.3) \qquad \begin{cases} v_t - \Delta v = 0 & \text{in } \Omega(u)\backslash K, \\ \quad\;\; v = 0 & \text{on } \Gamma(u), \\ \quad\;\; v = 1 & \text{in } K, \\ v(x,0) = v_0(x). \end{cases}$$

Then $v(x,t)$ is a viscosity solution of (1.1) in $B \times [0,T]$ with initial data $v(x,0) = v_0(x)$, and $u(x,t) = \int_0^t v(x,s)ds$.

Remark 2.11. The definition of the solution v of (2.3) must be clarified when $\Omega(u)$ is not smooth. Since u is continuous and $\Omega(u)$ is bounded at all times ([16, Lemma 3.6]) then the existence of solution of (2.3) is provided by Perron's method as

$$v = \sup\{w | w_t - \Delta w \le 0 \text{ in } \Omega(u), w \le 0 \text{ on } \Gamma(u), w \le 1 \text{ in } K, w(x,0) \le v_0(x)\}.$$

Note that v might be discontinuous on $\Gamma(u)$.

The coincidence of weak and viscosity solutions gives us a more general comparison principle.

Lemma 2.12 (cf. [16, Corollary 3.12]). *Let v^1 and v^2 be, respectively, a viscosity subsolution and supersolution of the Stefan problem (1.1) with continuous initial data $v_0^1 \le v_0^2$ and boundary data 1. In addition, suppose that v_0^1 (or v_0^2) satisfies condition (1.2). Then $v_\star^1 \le v^2$ and $v^1 \le (v^2)^\star$ in $\mathbb{R}^n \backslash K \times [0, \infty)$.*

2.4. **Rescaling.** We will use the following rescaling of solutions as in [17].

2.4.1. *For $n \ge 3$.* For $\lambda > 0$ we use the rescaling

$$v^\lambda(x,t) = \lambda^{\frac{n-2}{n}} v(\lambda^{\frac{1}{n}} x, \lambda t), \qquad u^\lambda(x,t) = \lambda^{-\frac{2}{n}} u(\lambda^{\frac{1}{n}} x, \lambda t).$$

If we define $K^\lambda := K/\lambda^{\frac{1}{n}}$ and $\Omega_0^\lambda := \Omega_0/\lambda^{\frac{1}{n}}$ then v^λ satisfies the problem

$$(2.4) \qquad \begin{cases} \lambda^{\frac{2-n}{n}} v_t^\lambda - \Delta v^\lambda = 0 & \text{in } \Omega(v^\lambda) \backslash K^\lambda, \\ \quad v^\lambda = \lambda^{\frac{n-2}{n}} & \text{on } K^\lambda, \\ \quad v_t^\lambda = g^\lambda(x) |Dv^\lambda|^2 & \text{on } \Gamma(v^\lambda), \\ v^\lambda(\cdot, 0) = v_0^\lambda, \end{cases}$$

where $g^\lambda(x) = g(\lambda^{\frac{1}{n}} x)$. And the rescaled u^λ satisfies the obstacle problem

$$(2.5)$$
$$\begin{cases} \qquad\qquad u^\lambda(\cdot, t) \in \mathcal{K}^\lambda(t), \\ (\lambda^{\frac{2-n}{n}} u_t^\lambda - \Delta u^\lambda)(\varphi - u^\lambda) \ge f(\lambda^{\frac{1}{n}} x)(\varphi - u^\lambda) & \text{a.e } (x,t) \in \mathbb{R}^n \times (0, \infty) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } \varphi \in \mathcal{K}^\lambda(t), \\ \qquad u^\lambda(x,0) = 0, \end{cases}$$

where $\mathcal{K}^\lambda(t) = \{\varphi \in H^1(\mathbb{R}^n), \varphi \ge 0, \varphi = \lambda^{\frac{n-2}{n}} t \text{ on } K^\lambda\}$.

Remark 2.13. We can take the admissible set $K^\lambda(t)$ as above due to the continuity with respect to the H^1 norm of all terms in the variational inequality and the fact that the variational solution u has a compact support in space at every time.

2.4.2. *For $n=2$.* For dimension $n = 2$, we use a different rescaling that preserves the singularity of logarithm, namely

$$(2.6) \qquad v^\lambda(x,t) = \log \mathcal{R}(\lambda) v(\mathcal{R}(\lambda) x, \lambda t), \qquad u^\lambda(x,t) = \frac{\log \mathcal{R}(\lambda)}{\lambda} u(\mathcal{R}(\lambda) x, \lambda t),$$

where $\mathcal{R}(\lambda)$ is the unique solution of $\mathcal{R}^2 \log \mathcal{R} = \lambda$, $\lim_{\lambda \to \infty} \mathcal{R}(\lambda) \to \infty$ (see [17] for more details). v^λ and u^λ satisfy rescaled problems analogous to (2.4) and (2.5). In particular, the term $\lambda^{(2-n)/n}$ in front of the time derivatives is replaced by $1/\log(\mathcal{R}(\lambda)) \to 0$ as $\lambda \to \infty$.

2.5. **Convergence of radially symmetric solutions.** We will recall the results
on the convergence of radially symmetric solutions of (1.1) as derived in [19]. First,
we collect some useful facts of radial solution of the Hele-Shaw problem and then use
a comparison to have the information of radial solution of the Stefan problem. The
radially symmetric solution of the Hele-Shaw problem in the domain $|x| \geq a, t \geq 0$
is a pair of functions $p(x,t)$ and $R(t)$, where p is of the form

$$(2.7) \qquad p(x,t) = \begin{cases} \dfrac{Aa^{n-2}\left(|x|^{n-2} - R^{n-2}(t)\right)_+}{a^{2-n} - R^{2-n}(t)}, & n \geq 3, \\[3mm] \dfrac{A\left(\log \frac{R(t)}{|x|}\right)_+}{\log \frac{R(t)}{a}}, & n = 2, \end{cases}$$

and $R(t)$ satisfies a certain algebraic equation (see [19] for details).

This solution satisfies the boundary conditions and initial conditions

$$(2.8) \qquad \begin{aligned} p(x,t) &= Aa^{2-n} && \text{for } |x| = a > 0, \\ p(x,t) &= 0 && \text{for } |x| = R(t), \\ R'(t) &= \frac{1}{L}|Dp| && \text{for } |x| = R(t), \\ R(0) &= b > a. \end{aligned}$$

Furthermore,

$$\lim_{t \to \infty} \frac{R(t)}{c_\infty t^{1/n}} = 1, \qquad c_\infty = \left(\frac{An(n-2)}{L}\right)^{1/n} \qquad \text{if } n \geq 3,$$

$$\lim_{t \to \infty} \frac{R(t)}{c_\infty \left(t/\log t\right)^{1/2}} = 1, \qquad c_\infty = 2\sqrt{A/L} \qquad \text{if } n = 2.$$

In dimension $n = 2$, we will also use $\lim\limits_{t \to \infty} \frac{\log R(t)}{\log t} = \frac{1}{2}$.

The radial solution of the Stefan problem satisfies the corresponding conditions
similar to (2.8) together with the initial data

$$(2.9) \qquad\qquad \theta(x,0) = \theta_0(|x|) \text{ if } |x| \geq a.$$

The following results were shown in [19].

Lemma 2.14 (cf. [19, Proposition 6.1]). *Let p and θ be radially symmetric solutions
to the Hele-Shaw problem and to the Stefan problem respectively, and let $\{|x| = R_p(t)\}, \{|x| = R_\theta(t)\}$ be the corresponding interfaces. If $R_p(0) > R_\theta(0), p(x,0) \geq \theta(x,0)$ and, moreover, $p(x,t) \geq \theta(x,t)$ on the fixed boundary, that is, for $|x| = a, t > 0$, then $p(x,t) \geq \theta(x,t)$ for all $|x| \geq a$ and $t \geq 0$.*

This immediately leads to an upper bound for the free boundary of radial solu-
tions of Stefan problem, see Corollary 6.2, Theorem 6.4, Theorem 7.1 in [19].

Lemma 2.15. *Let $\{|x| = R(t)\}$ be the free boundary of a radial solution to the
Stefan problem satisfying the corresponding conditions (2.8) and (2.9). There are
constants $C, T > 0$, such that, for all $t \geq T$,*

$$R(t) \leq Ct^{1/n}, n \geq 3, \quad \text{or} \quad R(t) \leq C(t/\log t)^{1/2}, n = 2.$$

Moreover, we have

$$\lim_{t \to \infty} \frac{R(t)}{t^{1/n}} = (An(n-2)/L)^{1/n}, n \geq 3, \quad \text{or} \quad \lim_{t \to \infty} \frac{R(t)}{(t/\log t)^{1/2}} = 2\sqrt{A/L}, n = 2.$$

The solution of the Stefan problem is bounded for all time.

Lemma 2.16 (cf. [19, Lemma 6.3]). *Let θ be a weak solution of the Stefan problem for $n \geq 2$. There is a constant $C > 0$ such that, for all $t > 0$, $0 \leq \theta(x,t) \leq C|x|^{2-n}$.*

Next, we define the solution of the Hele-Shaw problem with a point source, which will appear as the limit function in our convergence results,

$$(2.10) \qquad V(x,t) = V_{A,L}(x,t) = \begin{cases} A\left(|x|^{2-n} - \rho^{2-n}(t)\right)_+, & n \geq 3, \\ A\left(\log \frac{\rho(t)}{|x|}\right)_+, & n = 2, \end{cases}$$

where

$$\rho(t) = \rho_L(t) = R_\infty = \begin{cases} (An(n-2)t/L)^{1/n}, & n \geq 3, \\ (2At/L)^{1/2}, & n = 2. \end{cases}$$

It is the unique solution of the Hele-Shaw problem with a point source,

$$(2.11) \quad \begin{cases} \Delta v = 0 & \text{in } \Omega(v)\backslash\{0\}, \\ \lim_{|x|\to 0} \dfrac{v(x,t)}{|x|^{2-n}} = A, \quad n \geq 3, \quad \text{or} \quad \lim_{|x|\to 0} -\dfrac{v(x,t)}{\log(|x|)} = A, \quad n = 2, \\ v_t = \dfrac{1}{L}|Dv|^2 & \text{on } \partial\Omega(v), \\ v(x,0) = 0 & \text{in } \mathbb{R}^n\backslash\{0\}. \end{cases}$$

The asymptotic result for radial solutions of the Stefan problem follows from Theorem 6.5 and Theorem 7.2 in [19].

Theorem 2.17 (Far field limit). *Let θ be the radial solution of the Stefan problem satisfying the corresponding boundary and initial conditions (2.8), (2.9). Then*

$$(2.12) \qquad \lim_{t\to\infty} t^{(n-2)/n}|\theta(x,t) - V(x,t)| = 0$$

uniformly on sets of form $\{x \in \mathbb{R}^n : |x| \geq \delta t^{1/n}\}, \delta > 0$ if $n \geq 3$, and
$$(2.13)$$
$$\lim_{t\to\infty} \log\sqrt{\frac{2A}{L}}\mathcal{R}(t)\left|\theta(x,t) - \frac{A}{\log\sqrt{\frac{2A}{L}}\mathcal{R}(t)}\left(\log\sqrt{\frac{2A}{L}}\mathcal{R}(t) - \log|x|\right)_+\right| = 0$$

uniformly on sets of form $\{x \in \mathbb{R}^n : |x| \geq \delta\mathcal{R}(t)\}, \delta > 0$ if $n = 2$.

Proof. Following the proof of Theorem 6.5 in [19] with recalling that we assume $\theta = Aa^{2-n}$ for $|x| = a$, we immediately get the result for $n = 3$.

For $n = 2$, let $\mathcal{R}_1(t)$ be the solution of $\frac{\mathcal{R}_1^2}{2}\left(\log\mathcal{R}_1 - \frac{1}{2}\right) = \frac{At}{L}$ with $\lim_{t\to\infty} \frac{\mathcal{R}_1(t)}{\mathcal{R}(t)} = \sqrt{\frac{2A}{L}}$. Thus, we can replace $\mathcal{R}_1(t)$ in Theorem 7.2 in [19] by $\sqrt{\frac{2A}{L}}\mathcal{R}(t)$. $\qquad\square$

Finally, we can improve Theorem 2.17 to have the following convergence result for rescaled radial solutions of the Stefan problem which holds up to $t = 0$.

Lemma 2.18 (Convergence for radial case). *Let $\theta(x,t)$ be a radial solution of the Stefan problem satisfying the corresponding boundary and initial conditions (2.8) and (2.9). Then θ^λ converges locally uniformly to $V_{A,L}$ in the set $(\mathbb{R}^n\backslash\{0\})\times[0,\infty)$.*

Proof. We will prove the uniform convergence in the sets $Q = \{(x, t) : |x| \geq \varepsilon, 0 \leq t \leq T\}$ for some $\varepsilon, T > 0$ and use notation $V = V_{A,L}$. We consider the case $n \geq 3$ first. Set $\xi = \lambda^{1/n}x, \tau = \lambda t$ then an easy computation leads to $V(x, t) = \lambda^{(n-2)/n}V(\xi, \tau)$. Let $t_0 = \rho^{-1}(\varepsilon/2)$. We split the proof into two cases:

(a) When $0 \leq t \leq t_0$: Clearly from the formula, we have $V(x, t) = 0$ in $\{(x, t) : |x| \geq \varepsilon, 0 \leq t \leq t_0\}$. Besides, for λ large enough,

$$R_\lambda(t) = \frac{R(\lambda t)}{\lambda^{1/n}} \leq \frac{R(\lambda t_0)}{\lambda^{1/n}} < \rho(t_0) + \frac{\varepsilon}{2} = \varepsilon \text{ (due to Proposition 2.15).}$$

Thus, $\theta^\lambda = 0 = V$ in $\{(x, t) : |x| \geq \varepsilon, 0 \leq t \leq t_0\}$ for λ large enough.

(b) When $t_0 \leq t \leq T$, we have:

(2.14) $$|\theta^\lambda(x, t) - V(x, t)| = t^{(2-n)/n}\tau^{(n-2)/n}|\theta(\xi, \tau) - V(\xi, \tau)|$$

Since $t_0 \leq t \leq T$, $t^{(2-n)/n}$ is bounded. From Theorem 2.17, the right hand side of (2.14) converges to 0 uniformly in the sets $\{\xi \in \mathbb{R}^n : |\xi| \geq \delta\tau^{1/n}\} = \{x \in \mathbb{R}^n : |x| \geq \delta t^{1/n}\} \supset \{(x, t) : |x| \geq \varepsilon, t_0 \leq t \leq T\}$ for fixed ε and $\delta > 0$ small enough and thus we obtain the convergence for $n \geq 3$.

For $n = 2$, we argue similar as the case $n \geq 3$, but noting that $\lim_{\lambda \to \infty} \frac{\mathcal{R}(\tau)}{\mathcal{R}(\lambda)} = t^{1/2}$ together with Theorem 2.17. $\square$

2.6. **Some more results for viscosity solutions.** Following [17, 19], we also can state some results for viscosity solutions.

Lemma 2.19. *For $L = 1/m$ (resp. $L = 1/M$), with m, M as in (1), let $\theta(x, t)$ be the radial solution of Stefan problem (1.1) satisfying boundary conditions (2.8) and initial condition (2.9) with $g(x) = 1/L$ and a such that $B(0, a) \subset K$ (resp. $K \subset B(0, a)$). Then the function $\theta(x, t)$ is a viscosity subsolution (resp. supersolution) of the Stefan problem (1.1) in Q.*

Proof. The statement follows directly from properties of radially solutions and the fact that a classical solution is also a viscosity solution. $\square$

Using viscosity comparison principle, we also can get the same estimates for free boundary as in Proposition 2.15 and boundedness for a **general viscosity solution**.

Lemma 2.20. *Let v be a viscosity solution of (1.1). There exists $t_0 > 0$ and constant $C, C_1, C_2 > 0$ such that for $t \geq t_0$,*

$$C_1t^{1/n} < \min_{\Gamma_t(v)} |x| \leq \max_{\Gamma_t(v)} |x| < C_2t^{1/n} \qquad \text{if } n \geq 3,$$

$$C_1\mathcal{R}(t) < \min_{\Gamma_t(v)} |x| \leq \max_{\Gamma_t(v)} |x| < C_2\mathcal{R}(t) \qquad \text{if } n = 2,$$

and for $0 \leq t \leq t_0$, $\max_{\Gamma_t(v)} |x| < C_2$. Moreover, $0 \leq v(x, t) \leq C|x|^{2-n}$ for all $n \geq 2$.

Proof. Argue as in [17] with using Lemma 2.15 and Lemma 2.16 above. $\square$

We also have the near field limit and the asymptotic behavior result as in [19].

Theorem 2.21 (Near-field limit)**.** *The viscosity solution $v(x,t)$ of the Stefan problem (1.1) converges to the unique solution $P(x)$ of the exterior Dirichlet problem*

$$(2.15) \qquad \begin{cases} \Delta P = 0, & x \in \mathbb{R}^n \backslash K, \\ P = 1, & x \in \Gamma, \\ \lim_{|x| \to \infty} P(x) = 0 & \text{if } n \geq 3, \qquad or \quad P \text{ is bounded if } n = 2, \end{cases}$$

as $t \to \infty$ uniformly on compact subsets of $\overline{K^c}$.

Proof. See proof of Theorem 8.1 in [19]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.22 (cf. [19, Lemma 4.5])**.** *There exists a constant $C_* = C_*(K, n)$ such that the solution P of problem (2.15) satisfies $\lim_{|x| \to \infty} |x|^{n-2} P(x) = C_*$.*

3. Uniform convergence of variational solutions

3.1. **Limit problem and the averaging properties of media.** We first recall the limit variational problem as introduced in [17] (see [17, section 5] for derivation and properties). Let $U_{A,L}(x,t) := \int_0^t V_{A,L}(x,s)\, ds$. For given $A, L > 0$, [17, Theorem 5.1] yields that $U_{A,L}(x,t)$ is the unique solution of the limit obstacle problem

$$(3.1) \qquad \begin{cases} w \in \mathcal{K}_t, \\ a(w, \phi) \geq \langle -L, \phi \rangle, & \text{for all } \phi \in V, \\ a(w, \psi w) = \langle L, \psi w \rangle & \text{for all } \psi \in W, \end{cases}$$

where $\mathcal{K}_t = \left\{ \varphi \in \bigcap_{\varepsilon > 0} H^1(\mathbb{R}^n \backslash B_\varepsilon) \cap C(\mathbb{R}^n \backslash B_\varepsilon) : \varphi \geq 0, \lim_{|x| \to 0} \frac{\varphi(x)}{U_{A,L}(x,t)} = 1 \right\}$,

$$(3.2) \qquad V = \left\{ \phi \in H^1(\mathbb{R}^n) : \phi \geq 0, \phi = 0 \text{ on } B_\varepsilon \text{ for some } \varepsilon > 0 \right\},$$

$$(3.3) \qquad W = V \cap C^1(\mathbb{R}^n),$$

and

$$a_\Omega(u, v) := \int_\Omega Du \cdot Dv dx, \quad \langle u, v \rangle_\Omega := \int_\Omega uv dx.$$

We omit the set Ω in the notation if $\Omega = \mathbb{R}^n$.

We also recall the following application of the subadditive ergodic theorem.

Lemma 3.1 (cf. [15, Section 4, Lemma 7], see also [17])**.** *For given g satisfying (2), there exists a constant, denoted by $\langle 1/g \rangle$, such that if $\Omega \subset \mathbb{R}^n$ is a bounded measurable set and if $\{u^\varepsilon\}_{\varepsilon > 0} \subset L^2(\Omega)$ is a family of functions such that $u^\varepsilon \to u$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$, then*

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{1}{g(x/\varepsilon, \omega)} u^\varepsilon(x) dx = \int_\Omega \left\langle \frac{1}{g} \right\rangle u(x) dx \text{ a.e. } \omega \in A.$$

3.2. **Uniform convergence of rescaled variational solutions.** Now we are ready to prove the first main result, similar to Theorem 6.2 in [17].

Theorem 3.2. *Let u be the unique solution of variational problem (2.1) and u^λ be its rescaling. Let $U_{A,L}$ be the unique solution of limit problem (3.1) where $A = C_*$ as in Lemma 2.22, and $L = \langle 1/g \rangle$ as in Lemma 3.1. Then the functions u^λ converges locally uniformly to $U_{A,L}$ as $\lambda \to \infty$ on $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$.*

Proof. We argue as in [17]. Fix $T > 0$. By Lemma 2.20, we can bound $\Omega_t(u^\lambda)$ by $\Omega := B_\delta(0)$ for some $\delta > 0$, for all $0 \leq t \leq T$ and $\lambda > 0$. For some $\varepsilon > 0$, define $\Omega_\varepsilon := \Omega \backslash \overline{B(0, \varepsilon)}$, $Q_\varepsilon := \Omega_\varepsilon \times [0, T]$. We will prove the convergence in Q_ε.

Let v be the viscosity solution of the Stefan problem (1.1). We can find constants $0 < a < b$ such that $K \subset B_a(0)$ and $\overline{\Omega_0} \subset B_b(0)$. Set $L = 1/M$ and $A = \max v_0$. Choose radially symmetric smooth $\theta_0 \geq 0$ such that $\theta_0 \geq v_0$ on $\Omega_0 \setminus B_a(0)$ and $\theta_0 = 0$ on $\mathbb{R}^n \setminus B_b(0)$. The radial solution θ of the Stefan problem on $\mathbb{R}^n \setminus \overline{B_a(0)}$ with such parameters will be above v by the comparison principle. Thus, for λ large enough, the rescaled solutions satisfy

$$0 \leq v^\lambda \leq \theta^\lambda \text{ in } Q_{\varepsilon/2}.$$

On the other hand, by Lemma 2.18, θ^λ converges to $V_{A,L}$ as $\lambda \to \infty$ uniformly on $Q_{\varepsilon/2}$ and $V_{A,L}$ is bounded in $Q_{\varepsilon/2}$ and therefore for λ large enough so that $(B_a(0))^\lambda := \frac{B_a(0)}{\lambda^{1/n}} \subset B_{\varepsilon/2}(0)$,

$$(3.4) \qquad \|u_t^\lambda\|_{L^\infty(Q_{\varepsilon/2})} = \|v^\lambda\|_{L^\infty(Q_{\varepsilon/2})} \leq C(\varepsilon).$$

Since u^λ satisfies (2.5), we have

$$\Delta u^\lambda (\varphi - u^\lambda) \leq \left(\lambda^{(2-n)/n} u_t^\lambda - f(\lambda^{1/n} x) \right) (\varphi - u^\lambda) \text{ a.e for any } \varphi \in \mathcal{K}^\lambda(t).$$

As u_t^λ is bounded, u^λ satisfies the elliptic obstacle problem

$$\Delta u^\lambda (\varphi - u^\lambda) \leq \left(C\lambda^{(2-n)/n} - f(\lambda^{1/n} x) \right) (\varphi - u^\lambda)$$

a.e for any $\varphi \in \mathcal{K}^\lambda(t)$ such that $\varphi - u^\lambda \geq 0$.

Now we can use the standard regularity estimates for the obstacle problem (see [21, Proposition 2.2, chapter 5] for instance),

$$\|\Delta u^\lambda(\cdot, t)\|_{L^p(\Omega_{\varepsilon/2})} \leq \left\| C\lambda^{(2-n)/n} - \frac{1}{g^\lambda} \right\|_{L^p(\Omega_{\varepsilon/2})} \leq C_0 \text{ for all } 1 \leq p \leq \infty,$$

for all λ large so that also $\Omega_0^\lambda \subset B_{\varepsilon/2}(0)$. Using (3.4) and $u^\lambda(x, t) = \int_0^t v^\lambda(x, s)ds$, we conclude $\|u^\lambda(\cdot, t)\|_{L^p(\Omega_{\varepsilon/2})}$ is bounded uniformly in $t \in [0, T]$ and λ large.

Using elliptic interior estimate results for obstacle problem again (for example, [21, Theorem 2.5]), we can find constants $0 < \alpha < 1$ and C_2, independent of $t \in [0, T]$ and $\lambda \gg 1$, such that

$$\begin{aligned} \|u^\lambda(\cdot, t)\|_{W^{2,p}(\Omega_\varepsilon)} &\leq C_2, \\ \|u^\lambda(\cdot, t)\|_{C^{0,\alpha}(\Omega_\varepsilon)} &\leq C_2, \end{aligned} \quad \text{for all } 0 \leq t \leq T, \lambda \gg 1.$$

Moreover, using (3.4) again, we have $|u^\lambda(x, t) - u^\lambda(x, s)| \leq C_3|t - s|$. Thus u^λ is Hölder continuous in x with $0 < \alpha < 1$ and Lipschitz continuous in t. In particular, u^λ satisfies

$$\|u^\lambda\|_{C^{0,\alpha}(Q_\varepsilon)} \leq C_4(C_2, C_3) \text{ for all } \lambda \geq \lambda_0.$$

The argument for case $n = 2$ is similar.

By the Arzelà-Ascoli theorem, we can find a function $\bar{u} \in C((\mathbb{R}^n \setminus \{0\}) \times [0, \infty))$ and a subsequence $\{u^{\lambda_k}\} \subset \{u^\lambda\}$ such that

$$u^{\lambda_k} \to \bar{u} \text{ locally uniformly on } (\mathbb{R}^n \setminus \{0\}) \times [0, \infty) \text{ as } k \to \infty,$$

Due to the compact embedding of H^2 in H^1, we have, $u^{\lambda_k}(\cdot, t) \to \bar{u}(\cdot, t)$ strongly in $H^1(\Omega_\varepsilon)$ for all $t \geq 0$, $\varepsilon > 0$.

To finish the proof, we need to show that the function $\bar{u}$ is the solution of limit problem (3.1) and then by the uniqueness of the limit problem, we deduce that the convergence is not restricted to a subsequence.

Lemma 3.3 (cf. [17, Lemma 6.3]). *For each $t \geq 0$, $\bar{w} := \bar{u}(\cdot, t)$ satisfies*

$$(3.5) \qquad\qquad a(\bar{w}, \phi) \geq \langle -L, \phi \rangle \text{ for all } \phi \in V,$$

$$(3.6) \qquad\qquad a(\bar{w}, \psi\bar{w}) = \langle -L, \psi\bar{w} \rangle \text{ for all } \psi \in W,$$

where $L = \langle 1/g \rangle$ as in Lemma 3.1 and V, W as in (3.2) and (3.3).

Proof. Consider $n \geq 3$. Following the techniques in [17], fix $t \in [0, T]$ and denote $w^k := u^{\lambda_k}(\cdot, t)$. Take $\phi \in V$ first. Analogously to Remark 2.13, we only need to prove the inequality for functions ϕ with compact support, the conclusion for general function ϕ in V will follow by the continuity of all terms in the inequality. There exists $k_0 > 0$ such that for all $k \geq k_0$, $\Omega_0^{\lambda_k} \subset B_\varepsilon(0)$ and $\phi = 0$ on $B_\varepsilon(0)$. Set $\varphi^k = \phi + w^k \in \mathcal{K}^{\lambda_k}(t)$. Substitute the function φ^k into the rescaled equation (2.5) and integrate both sides and integrate by parts, which yields

$$a(w^{\lambda_k}, \phi) \geq -\lambda_k^{(2-n)/n} \left\langle u_t^{\lambda_k}(\cdot, t), \phi \right\rangle + \left\langle -\frac{1}{g^{\lambda_k}}, \phi \right\rangle.$$

The linear functional $w \mapsto a(w, \phi)$ is bounded in H^1. Recalling Lemma 3.1 and that $u_t^{\lambda_k}$ is bounded, since $w^k \to \bar{w}$ strongly in H^1 as $k \to \infty$, we can send $\lambda_k \to \infty$ and obtain (3.5).

Now take $\psi \in W$. As above, we assume that ψ has compact support, and without loss of generality we can also assume that $0 \leq \psi \leq 1$, $\psi = 0$ on $B_\varepsilon(0)$ (otherwise consider $\frac{\psi}{\max_{\mathbb{R}^n} \psi}$ instead). Take k_0 such that $\Omega_0^{\lambda_k} \subset B_\varepsilon(0)$ for all $k \geq k_0$. Since $\psi \in W$ then $\psi\bar{w} \in V$. As above we have $a(\bar{w}, \psi\bar{w}) \geq \langle -L, \psi\bar{w} \rangle$. Moreover, consider $\varphi^k = (1 - \psi)w^k \in \mathcal{K}^{\lambda_k}(t)$, $k \geq k_0$. Then,

$$a(w^k, \psi w^k) = -a(w^k, \varphi^k - w^k) \leq \left\langle -\frac{1}{g^{\lambda_k}}, \psi w^k \right\rangle - \lambda_k^{(2-n)/n} \left\langle u_t^{\lambda_k}(\cdot, t), \psi w^k \right\rangle.$$

Again using Lemma 3.1, boundedness in $L^\infty(\mathbb{R}^n)$ of w^k and $u_t^{\lambda_k}$, the lower semi-continuity in H^1 of the map $w \mapsto a(w, \psi w)$, and the fact that $w^k \to \bar{w}$ strongly in H^1 as $k \to \infty$ we can conclude the equality (3.6).

Again, $n = 2$ is similar. □

Finally, the next lemma establishes that the singularity of $\bar{u}$ as $|x| \to 0$ is correct.

Lemma 3.4 (cf. [17, Lemma 6.4]). *We have*

$$\lim_{|x| \to 0} \frac{\bar{u}(x, t)}{U_{C_*, L}}(x, t) = 1$$

for every $t \geq 0$, where $L = \langle 1/g \rangle$ as in Lemma 3.1 and C_* as in Lemma 2.22.

Proof. The proof follows from the proof of [17, Lemma 6.4] since the solutions of the Stefan problem have the same near field limit (Lemma 2.15) as the Hele-Shaw solutions. □

This finishes the proof of Theorem 3.2 . □

4. Uniform convergence of rescaled viscosity solutions and free boundaries

In this section, we will deal with the convergence of v^λ and their free boundaries. Let v be a viscosity solution of the Stefan problem (1.1) and v^λ be its rescaling. Let $V = V_{C_*, L}$ be the solution of Hele-Shaw problem with a point source as in (2.10), where C_* is the constant of Lemma 2.22 and $L = \langle 1/g \rangle$ as in Lemma 3.1.

We define the half-relaxed limits in $\{|x| \neq 0, t \geq 0\}$:

$$v^*(x,t) = \limsup_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y,s), \qquad v_*(x,t) = \liminf_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y,s),$$

Remark 4.1. V is continuous in $\{|x| \neq 0, t \geq 0\}$, therefore $V_* = V = V^*$.

To complete Theorem 1.1, we prove a result similar to [17, Theorem 7.1.]

Theorem 4.2. *The rescaled viscosity solution v^λ of the Stefan problem (1.1) converges locally uniformly to $V = V_{C_*, \langle 1/g \rangle}$ in $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$ as $\lambda \to \infty$ and*

$$v_* = v^* = V.$$

Moreover, the rescaled free boundary $\{\Gamma(v^\lambda)\}_\lambda$ converges to $\Gamma(V)$ locally uniformly with respect to the Hausdorff distance.

To prepare for the proof of Theorem 4.2, we need to collect some results which are similar to the ones in [16] and [17] with some adaptations to our case. All the results for $n \geq 3$ we have in this section can be obtained for $n = 2$ by using limit $\frac{1}{\log \mathcal{R}(\lambda)} \to 0$ as $\lambda \to \infty$. Thus, from here on we only consider case $n \geq 3$, the results for $n = 2$ are omitted.

4.1. Some necessary technical results.

Lemma 4.3 (cf. [16, Lemma 3.9]). *The viscosity solution v of the Stefan problem (1.1) is strictly positive in $\Omega(u)$, satisfies $\Omega(v) = \Omega(u)$ and $\Gamma(v) = \Gamma(u)$.*

Lemma 4.4. *Let v^λ be a viscosity solution of the rescaled problem (2.4). Then $v^*(\cdot, t)$ is subharmonic in $\mathbb{R}^n \setminus \{0\}$ and $v_*(\cdot, t)$ is superharmonic in $\Omega_t(v_*) \backslash \{0\}$ in viscosity sense.*

Proof. This follows from a standard viscosity solution argument using test functions, see for instance [11]. $\square$

The behavior of functions v^*, v_* at the origin and their boundaries can be established by following the arguments in [17] and [16].

Lemma 4.5 (v^* and v_* behave as V at the origin). *The functions v^*, v_* have a singularity at 0 with:*

$$(4.1) \qquad \lim_{|x| \to 0+} \frac{v_*(x,t)}{V(x,t)} = 1, \qquad \lim_{|x| \to 0+} \frac{v^*(x,t)}{V(x,t)} = 1, \text{ for each } t > 0.$$

Proof. See [17, Lemma 7.4]. $\square$

Lemma 4.6 (cf. [16, Lemma 5.4]). *Suppose that $(x_k, t_k) \in \{u^{\lambda_k} = 0\}$ and $(x_k, t_k, \lambda_k) \to (x_0, t_0, \infty)$. Then:*

a) $U(x_0, t_0) = 0$,
b) If $x_k \in \Gamma_{t_k}(u^{\lambda_k})$ then $x_0 \in \Gamma_{t_0}(U)$,

where $U = U_{C_*, L}$ is the limit function in Theorem 3.2.

Proof. See proof of [16, Lemma 5.4]. $\qquad \square$

The rest of the convergence proof in [17] relies on the monotonicity of the solutions of the Hele-Shaw problem in time. Since the Stefan problem lacks this monotonicity, we will show that sufficiently regular initial data satisfies a weak monotonicity below. The convergence result for general initial data will then follow by the uniqueness of the limit and the comparison principle.

Lemma 4.7. *Suppose that v_0 satisfies* (1.2). *Then there exist $C \geq 1$ independent of x and t such that*

$$(4.2) \qquad v_0(x) \leq Cv(x, t) \ in \ \mathbb{R}^n \backslash K \times [0, \infty).$$

Proof. Let $\gamma_1 := \min_{\partial \Omega_0} |Dv_0|, \gamma_2 := \max_{\partial \Omega_0} |Dv_0|$. Note that $0 < \gamma_1 \leq \gamma_2 < \infty$. For given $\varepsilon > 0$, let w be the solution of boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega_0 \backslash K, \\ w = \varepsilon & \text{on } K, \\ w = 0 & \text{on } \Omega_0^c. \end{cases}$$

For x close to $\partial \Omega_0$ we have $v_0(x) \geq \frac{\gamma_1}{2} \text{dist}(x, \partial \Omega_0)$. Since $\gamma_1 > 0$, $v_0 > 0$ in Ω_0 and $\partial \Omega_0$ has a uniform ball condition, we can choose $\varepsilon > 0$ small enough such that $w \leq v_0$ in $\mathbb{R}^n \backslash K$. By Hopf's Lemma, $\gamma_w := \min_{\partial \Omega_0} |Dw| > 0$. It is clear that w is a classical subsolution of the Stefan problem (1.1) and the comparison principle yields

$$(4.3) \qquad w \leq v \text{ in } (\mathbb{R}^n \backslash K) \times [0, \infty).$$

Now assume that (4.2) does not hold, that is, for every $k \in \mathbb{N}$, there exists $(x_k, t_k) \in \mathbb{R}^n \backslash K \times [0, \infty)$ such that

$$(4.4) \qquad \frac{1}{k} v_0(x_k) > v(x_k, t_k).$$

Clearly $x_k \in \Omega_0$. $\{t_k\}$ is bounded by Theorem 2.21 since v_0 is bounded. Therefore, there exists a subsequence (x_{k_l}, t_{k_l}) and a point (x_0, t_0) such that $(x_{k_l}, t_{k_l}) \to (x_0, t_0)$. Since v_0 is bounded, we get $v(x_0, t_0) \leq 0$ and thus $x_0 \in \partial \Omega_0$ by (4.3). Consequently, for k_l large enough,

$$w(x_{k_l}) \geq \frac{1}{2} \gamma_w \text{dist}(x_{k_l}, \partial \Omega_0) = \left(\frac{\gamma_w}{4\gamma_2} \right) 2\gamma_2 \text{dist}(x_{k_l}, \partial \Omega_0) \geq \frac{\gamma_w}{4\gamma_2} v_0(x_{k_l}).$$

Combine this with (4.3) and (4.4) to obtain

$$\frac{1}{k_l} v_0(x_{k_l}) > \frac{\gamma_w}{4\gamma_2} v_0(x_{k_l})$$

for every k_l large enough, which yields a contradiction since $v_0(x_{k_l}) > 0$. $\qquad \square$

Some of the following lemmas will hold under the condition (4.2).

Lemma 4.8. *Let u be the solution of the variational problem* (2.1), *and v be the associated viscosity solution of the Stefan problem, and suppose that* (4.2) *holds. Then*

$$(4.5) \qquad u(x, t) \leq Ctv(x, t).$$

Proof. The statement follows from checking that $\tilde{u} := Ctv$ is a supersolution of the heat equation in $\Omega(u)$ and the classical comparison principle. Indeed, $\tilde{u}_t - \Delta \tilde{u} = Cv + Ct(v_t - \Delta v) \geq v_0 \geq f = u_t - \Delta u$ in $\Omega(u)$ by (4.2). $\qquad \square$

Lemma 4.9 (cf. [16, Lemma 5.5]). *The function v_* satisfies $\Omega(V) \subset \Omega(v_*)$. In particular $v_* \geq V$.*

Proof. Assume that the inclusion does not hold, there exists $(x_0, t_0) \in \Omega(V)$ and $v_*(x_0, t_0) = 0$. By (4.2) and Lemma 4.8, there exists $C > 1$ such that $u(x, t) \leq Ctv(x, t)$. This inequality is preserved under the rescaling, $u^\lambda(x, t) \leq Ctv^\lambda(x, t)$ in $(\mathbb{R}^n \backslash K^\lambda) \times [0, \infty)$. Taking $\liminf^*$ of both sides gives the contradiction $0 < U(x_0, t_0) \leq Ct_0 v_*(x_0, t_0) = 0$.

The inequality $v_* \geq V$ follows from the elliptic comparison principle as v_* is superharmonic in $\Omega(v_*) \setminus \{0\}$ by Lemma 4.4 and behaves as V at the origin by Lemma 4.5. $\square$

Lemma 4.10. *There exists constant $C > 0$ independent of λ such that for every $x_0 \in \overline{\Omega_{t_0}(u^\lambda)}$ and $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ for some r, for every λ large enough we have*

$$\sup_{x \in \overline{B_r(x_0)}} u^\lambda(x, t_0) > Cr^2.$$

Proof. Follow the arguments in [15, Lemma 3.1] while noting that since u_t^λ is bounded then, for λ large enough, u^λ is a strictly subharmonic function in $\Omega_{t_0}(u^\lambda) \setminus \overline{\Omega_0^\lambda}$. $\square$

Corollary 4.11. *There exists a constant $C_1 = C_1(n, M, \lambda_0)$ such that if $(x_0, t_0) \in \Omega(v^\lambda)$ and $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ and $\lambda \geq \lambda_0$, we have*

$$\sup_{B_r(x_0)} v^\lambda(x, t_0) \geq \frac{C_1 r^2}{t_0}.$$

Proof. The inequality follows directly from Lemma 4.8 and Lemma 4.10. $\square$

Lemma 4.12 (cf. [16, Lemma 5.6 ii]). *We have the following inclusion:*

$$\Gamma(v^*) \subset \Gamma(V).$$

Proof. Argue as in [16, Lemma 5.6 ii] together with using Lemma 4.6 and Lemma 4.10 above. $\square$

Now we are ready to prove Theorem 4.2.

4.2. **Proof of Theorem 4.2.**

Proof. **Step 1**. We prove the convergence of viscosity solutions and the free boundaries under the conditions (1.2) and (4.2) first.

Lemma 2.20 yields that $\Omega_t(v^*)$ is bounded at all time $t > 0$. Since $\Omega(V)$ is simply connected set, Lemma 4.12 implies that

$$\overline{\Omega(v^*)} \subset \overline{\Omega(V)} \subset \Omega(V_{C^*+\varepsilon, L}) \text{ for all } \varepsilon > 0.$$

We see from Lemma 4.4, $v^*(\cdot, t)$ is a subharmonic function in $\mathbb{R}^n \setminus \{0\}$ for every $t > 0$ and $\lim_{|x| \to 0} \frac{v^*(x,t)}{V(x,t)} = 1$ for all $t \geq 0$ by Lemma 4.5, comparison principle yields $v^*(x, t) \leq V_{C_*+\varepsilon, L}(x, t)$ for every $\varepsilon > 0$.

By Lemma 4.9, $V(x, t) \leq v_*$ and letting $\varepsilon \to 0^+$ we obtain by continuity

$$V(x, t) \leq v_*(x, t) \leq v^*(x, t) \leq V(x, t).$$

Therefore, $v_* = v^* = V$ and in particular, $\Gamma(v_*) = \Gamma(v^*) = \Gamma(V)$.

Now we need to show the uniform convergence of the free boundaries with respect to the Hausdorff distance. Fix $0 < t_1 < t_2$ and denote:

$$\Gamma^\lambda := \Gamma(v^\lambda) \cap \{t_1 \le t \le t_2\}, \qquad \Gamma^\infty := \Gamma(V) \cap \{t_1 \le t \le t_2\},$$

a δ-neighborhood of a set A in $\mathbb{R}^n \times \mathbb{R}$ is

$$U_\delta(A) := \{(x,t) : \text{dist}((x,t), A) < \delta\}.$$

We need to prove that for all $\delta > 0$, there exists $\lambda_0 > 0$ such that:

(4.6) $\qquad\qquad \Gamma^\lambda \subset U_\delta(\Gamma^\infty) \quad \text{and} \quad \Gamma^\infty \subset U_\delta(\Gamma^\lambda), \qquad \forall \lambda \ge \lambda_0.$

We prove the first inclusion in (4.6) by contradiction. Suppose therefore that we can find a subsequence $\{\lambda_k\}$ and a sequence of points $(x_k, t_k) \in \Gamma^{\lambda_k}$ such that $\text{dist}((x_k, t_k), \Gamma^\infty) \ge \delta$. Since Γ^λ is uniformly bounded in λ by Lemma 2.20, there exists a subsequence $\{(x_{k_j}, t_{k_j})\}$ which converge to a point (x_0, t_0). By Lemma 4.6, $(x_0, t_0) \in \Gamma(U) = \Gamma(V)$. Moreover, since $t_1 \le t_{k_j} \le t_2$ then $t_1 \le t_0 \le t_2$ and therefore, $(x_0, t_0) \in \Gamma^\infty$, a contradiction.

The proof of the second inclusion in (4.6) is more technical. We prove a pointwise result first. Suppose that there exists $\delta > 0, (x_0, t_0) \in \Gamma^\infty$ and $\{\lambda_k\}, \lambda_k \to \infty$, such that $\text{dist}((x_0, t_0), \Gamma^{\lambda_k}) \ge \dfrac{\delta}{2}$ for all k. Then there exists $r > 0$ such that $D_r(x_0, t_0) := B(x_0, r) \times [t_0 - r, t_0 + r]$ satisfies either:

(4.7) $\qquad\qquad\qquad D_r(x_0, t_0) \subset \{v^{\lambda_k} = 0\}$ for all k,

or after passing to a subsequence,

(4.8) $\qquad\qquad\qquad D_r(x_0, t_0) \subset \{v^{\lambda_k} > 0\}$ for all k.

If (4.7) holds, clearly $V = v_* = 0$ in $D_r(x_0, t_0)$ which is in a contradiction with the assumption that $(x_0, t_0) \in \Gamma^\infty$.

Thus we assume that (4.8) holds. In $D_r(x_0, t_0)$, v^{λ_k} solves the heat equation $\lambda^{(2-n)/n} v_t^{\lambda_k} - \Delta v^{\lambda_k} = 0$. Set

$$w^k(x,t) := v^{\lambda_k}(x, \lambda_k^{(2-n)/n} t)$$

then $w^k > 0$ in $D_r^w(x_0, t_0) := B(x_0, r) \times [\lambda_k^{(n-2)/n}(t_0 - r), \lambda_k^{(n-2)/n}(t_0 + r)]$ and w^k satisfies $w_t^k - \Delta w^k = 0$ in $D_r^w(x_0, t_0)$. Since $\lambda_k^{(n-2)/n} \frac{r}{2} \to \infty$ as $k \to \infty$, by Harnack's inequality for the heat equation, for fixed $\tau > 0$ there exists a constant $C_1 > 0$ such that for each $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$ and λ_k such that $\tau < \lambda_k^{(n-2)/n} \frac{r}{4}$ we have

$$\sup_{B(x_0, r/2)} w^k(\cdot, \lambda_k^{(n-2)/n} t - \tau) \le C_1 \inf_{B(x_0, r/2)} w^k(\cdot, \lambda_k^{(n-2)/n} t).$$

This inequality together with Corollary 4.11 yields:

$$\frac{C_2 r^2}{t - \lambda_k^{(2-n)/n} \tau} \le \sup_{B(x_0, r/2)} v^{\lambda_k}(\cdot, t - \lambda_k^{(2-n)/n} \tau) \le C_1 \inf_{B(x_0, r/2)} v^{\lambda_k}(\cdot, t)$$

for all $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, $\lambda_k \ge \lambda_0$ large enough, where C_2 only depends on n, M, λ_0. Taking the limit when $\lambda_k \to \infty$, the uniform convergence of $\{v^{\lambda_k}\}$ to V gives $V > 0$ in $B(x_0, \frac{r}{2}) \times [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, which is a contradiction with $(x_0, t_0) \in \Gamma^\infty \subset \Gamma(V)$.

We have proved that every point of Γ^∞ belongs to all $U_{\delta/2}(\Gamma^\lambda)$ for sufficiently large λ. Therefore the second inclusion in (4.6) follows from the compactness of Γ^∞.

This concludes the proof of Theorem 4.2 when (4.2) holds.

Step 2. For general initial data, we will find upper and lower bounds for the initial data for which (4.2) holds, and use the comparison principle. For instance, assume that $v_0 \in C(\mathbb{R}^n)$, $v_0 \geq 0$, such that $\operatorname{supp} v_0$ is bounded, $v_0 = 1$ on K.

Choose smooth bounded domains Ω_0^1, Ω_0^2 such that $K \subset \Omega_0^1 \subset \overline{\Omega_0^1} \subset \operatorname{supp} v_0 \subset \Omega_0^2$. Let v_0^1, v_0^2 be two functions satisfying (1.2) with positive domains Ω_0^1, Ω_0^2, respectively, and $v_0^1 \leq v_0 \leq v_0^2$. If necessary, that is, when v_0 is not sufficiently regular at ∂K, we may perturb the boundary data for v_0^1, v_0^2 on K as $1 - \varepsilon$ and $1 + \varepsilon$, respectively, for some $\varepsilon \in (0, 1)$.

Let v_1, v_2 be respectively the viscosity solution of the Stefan problem (1.1) with initial data v_0^1, v_0^2. By the comparison principle, we have $v_1 \leq v \leq v_2$ and after rescaling $v_1^\lambda \leq v^\lambda \leq v_2^\lambda$. By Step 1, we see that $v_1^\lambda \to V_{C_*, 1-\varepsilon, L}$ and $v_2^\lambda \to V_{C_*, 1+\varepsilon, L}$. Since $C_{*, 1 \pm \varepsilon} \to C_*$ as $\varepsilon \to 0$ by [19, Lemma 4.5], we deduce the local uniform convergence of $v^\lambda \to V = V_{C_*, L}$.

The convergence of free boundaries follows from the ordering $\Omega(v_1) \subset \Omega(v) \subset \Omega(v_2)$ and the convergence of free boundaries of $V_{C_*, 1 \pm \varepsilon, L}$ to the free boundary of $V_{C_*, L}$ locally uniformly with respect to the Hausdorff distance. $\square$

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